

CONSTANT Q -CURVATURE METRICS NEAR THE HYPERBOLIC METRIC

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ABSTRACT. Let (M, g) be a Poincaré-Einstein manifold with a smooth defining function. In this note, we prove that there are infinitely many asymptotically hyperbolic metrics with constant Q -curvature in the conformal class of an asymptotically hyperbolic metric close enough to g . These metrics are parametrized by the elements in the kernel of the linearized operator of the prescribed constant Q -curvature equation. A similar analysis is applied to a class of fourth order equations arising in spectral theory.

1. INTRODUCTION

In this note we will discuss the prescribed constant Q -curvature problem for asymptotically hyperbolic manifolds. We obtain the existence of a family of constant Q -curvature metrics in a small neighborhood of any Poincaré-Einstein metric, parametrized by elements in the null space of the linearized operator L in (1.3). Much of the analysis follows from Mazzeo's microlocal analysis method for elliptic edge operators. Results in this setting have been proved for the scalar curvature equation, see [1].

For $n \geq 4$, a natural conformal invariant and the corresponding conformal covariant operator are the Q -curvature and the fourth order Paneitz operator. Let Ric_g and R_g be the Ricci curvature and the scalar curvature of (M, g) . The Q -Curvature and the Paneitz operator are defined as follows,

$$Q_g = \begin{cases} -\frac{1}{12}(\Delta_g R_g - R_g^2 + 3|\text{Ric}_g|^2), & n = 4, \\ -\frac{2}{(n-2)^2}|\text{Ric}_g|^2 + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}R_g^2 - \frac{1}{2(n-1)}\Delta_g R_g, & n \geq 5. \end{cases}$$

$$P_g(\varphi) = \begin{cases} \Delta_g^2 \varphi - \text{div}(\frac{2}{3}R_g g - 2\text{Ric}_g)d\varphi, & n = 4, \\ \Delta_g^2 \varphi - \text{div}_g(a_n R_g g - b_n \text{Ric}_g)\nabla_g \varphi + \frac{n-4}{2}Q_g \varphi, & n \geq 5, \end{cases}$$

where $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$, $b_n = \frac{4}{n-2}$, $\text{div}_g X = \nabla_i X^i$ for any smooth vector field X , and φ is any smooth function on M .

Let $\tilde{g} = \rho g$, with ρ a positive function on M , so that

$$\rho = \begin{cases} e^{2u}, & n = 4, \\ u^{\frac{4}{n-4}}, & n \geq 5. \end{cases}$$

The Q -curvature has the following transformation,

$$\begin{aligned} P_g u + 2Q_g &= 2Q_{\tilde{g}} e^{4u}, \quad n = 4, \\ P_g u &= \frac{n-4}{2} Q_{\tilde{g}} u^{\frac{n+4}{n-4}}, \quad n > 4. \end{aligned}$$

Note that Paneitz operator satisfies the following conformal covariance property for $\varphi \in C^\infty(M)$,

$$\begin{aligned} P_{\tilde{g}} \varphi &= e^{-4u} P_g \varphi, \quad n = 4, \\ P_{\tilde{g}}(\varphi) &= u^{-\frac{n+4}{n-4}} P_g(u \varphi), \quad n > 4. \end{aligned}$$

We want to find a function u so that the metric \tilde{g} satisfies $Q_{\tilde{g}} = f$ for a given function f . For the prescribed Q -curvature problem on closed manifold M of dimension four there are many results, see [3], [6], [10], [11]. In [21] a boundary value problem for this problem is solved. A flow approach is performed in [2], see also [4]. For $n \geq 5$, see [5], [20] and [24].

There are some interesting results for complete non-compact manifolds. For Euclidean space \mathbb{R}^n , $n \geq 4$, see [14] and [23]. In [9], using shooting method, the authors proved that there are infinitely many complete metrics with constant Q -curvature in the conformal class of the Poincaré disk with dimension $n \geq 5$, which are radially symmetric ODE solutions to the initial value problem parametrized by distinct given initial data at the origin. It is not difficult to prove that similar results hold for $n = 4$. Mazzeo pointed out that there should be a more general result of this type. In this paper, we solve a perturbation problem in the setting of asymptotically hyperbolic metrics close to a Poincaré-Einstein metric. To give a precise statement we first need some definitions.

Definition 1.1. *Let M be a smooth manifold of dimension n , with smooth boundary ∂M of dimension $n - 1$. Let g be a complete metric on $M = \text{Int}(\overline{M})$. We say that g is asymptotically hyperbolic if there exists a smooth function x on \overline{M} , with the property that $x > 0$ in M , and $x = 0$ on ∂M , so that the metric $h = x^2 g$ is well defined and smooth on \overline{M} , and $|dx|_h|_{\partial M} = 1$. Here x is called a defining function of g . Moreover, if $h \in C^{k,\alpha}$, for some positive integer k , we say that g is asymptotically hyperbolic of order $C^{k,\alpha}$. If g is also Einstein, we call g a Poincaré-Einstein metric, and (M, g) a Poincaré-Einstein manifold.*

Let (M^n, g) be an asymptotically hyperbolic manifold of dimension n , with x as its smooth defining function. Actually, we can choose x so that $|dx|_h = 1$ in a neighborhood of ∂M , see [7], and here for simple notation we always choose a defining function in this sense except in Section 4. We will mainly focus on the asymptotic behavior of the metric near ∂M , which is a local discussion. Let y be local coordinates on ∂M . In a neighborhood of ∂M in \overline{M} , we introduce the local coordinates in the following way: $(x, y) \in [0, \varepsilon) \times \partial M$ represent the point moving from the point on ∂M with local coordinate y , along the geodesic which is the integral curve of $\nabla_h x$ for a length x in the

metric h . In local coordinates (x, y) ,

$$h = x^2 g = dx^2 + \sum_{i,j=1}^{n-1} h_{ij} dy^i dy^j.$$

For convenience, let $\tilde{g} = \rho g$, with ρ a positive function on M , so that

$$\rho = \begin{cases} e^{2u}, & n = 4, \\ (1 + u)^{\frac{4}{n-4}}, & n \geq 5. \end{cases}$$

Let the operator \mathcal{E} be defined by

$$(1.1) \quad \mathcal{E}(u) = \begin{cases} P_g u + 2Q_g - 2Q_{\tilde{g}} e^{4u}, & \text{for } n = 4, \\ P_g(1 + u) - \frac{n-4}{2} Q_{\tilde{g}} (1 + u)^{\frac{n+4}{n-4}}, & \text{for } n \geq 5. \end{cases}$$

To solve the prescribed Q -curvature problem amounts to finding a solution to

$$(1.2) \quad \mathcal{E}(u) = 0.$$

We define the linear operator $L = L_g$ as follows,

$$(1.3) \quad L(u) = \begin{cases} P_g u - 8Q_g u, & n = 4, \\ P_g u - \frac{n+4}{2} Q_g u, & n \geq 5. \end{cases}$$

Let (x, y) be the local coordinates of M near the boundary defined as above. Let \mathcal{V}_e be the collection of the smooth vector fields on \overline{M} , which restricted in the neighborhood of ∂M , are generated by $\{x\partial_x, x\partial_{y^1}, \dots, x\partial_{y^{n-1}}\}$ with smooth coefficients on \overline{M} .

Next we introduce the weighted spaces that we will be using. First, the weighted Sobolev spaces,

$$x^\delta H_e^m(M, \Omega^{\frac{1}{2}}) = \{u = x^\delta v : V_1 \dots V_j v \in L^2(M, \Omega^{\frac{1}{2}}), \forall j \leq m, V_i \in \mathcal{V}_e\},$$

where $m \in \mathbb{N}$, $\delta \in \mathbb{R}$, and $\Omega^{\frac{1}{2}} = \sqrt{dx dy}$ is the half-density. We also introduce the weighted Hölder space,

$$x^\delta \Lambda^{m, \alpha} = x^\delta \Lambda^{m, \alpha}(M, \Omega^{\frac{1}{2}}) = \{u = x^\delta v \sqrt{dx dy} : V_1 \dots V_j v \in \Lambda^{0, \alpha}, \forall j \leq m, V_i \in \mathcal{V}_e\},$$

with $m \in \mathbb{N}$, $\delta \in \mathbb{R}$, and $0 < \alpha < 1$, where $\Lambda^{0, \alpha}(M)$ is the space of half-densities $u = v \sqrt{dx dy}$ such that

$$\|v\|_{\Lambda^{0, \alpha}(M)} = \sup |v| + \sup \frac{(x + \tilde{x})^\alpha |v(x, y) - v(\tilde{x}, \tilde{y})|}{|x - \tilde{x}|^\alpha + |y - \tilde{y}|^\alpha} < \infty.$$

We will use the norm

$$\|u\|_{x^\delta \Lambda^{k, \alpha}(M)} = \sum_{m=0}^k \sum_{|\gamma|=m} \|\partial_e^\gamma v\|_{0, \alpha},$$

with $\partial_e \in \mathcal{V}_e$ and $u = x^\delta v$.

In this paper, we always assume $n \geq 4$ to be the dimension of M . With these definitions, we can now state our main result:

Theorem 1.2. Let $(B_1^n(0), g)$ be the Poincaré disk, of dimension $n \geq 4$. Also, let x be a smooth defining function of g . Let L be the linear operator defined in (1.3). Let ν be a constant in the interval $(0, \frac{n-1}{2})$. Then,

- i) Kernel of L in the weighted space $x^\nu \Lambda^{4,\alpha}(M)$ for $0 < \alpha < 1$ is of infinite dimension. Also, L is surjective. For each element v in the kernel $\ker(L)$ with sufficiently small norm, and a given function $Q_{\tilde{g}} \in \Lambda^{0,\alpha}(M, \sqrt{dx dy})$ so that $(Q_{\tilde{g}} - Q_g)$ is in $x^\nu \Lambda^{0,\alpha}(M, \sqrt{dx dy})$ with the norm $\|Q_{\tilde{g}} - Q_g\|_{x^\nu \Lambda^{0,\alpha}}$ small enough, there exists a unique solution $u \in x^\nu \Lambda^{4,\alpha}(M)$ to the problem (1.2), so that the projection P_1 (see in Theorem 1.5) of u onto $\ker(L)$ is given by v .
- ii) Moreover, if $Q_{\tilde{g}} = Q_g$, u has the expansion near the boundary

$$(1.4) \quad u(x, y) \sim (u_{00}(y)x^{\frac{n-1}{2}+i\beta} + u_{10}(y)x^{\frac{n-1}{2}-i\beta}) + o(x^{\frac{n-1}{2}}),$$

with $\beta = \frac{\sqrt{n^2+2n-9}}{2}$ and $i = \sqrt{-1}$, where u_{00} and u_{10} are generally distributions of negative order. Also, u will have the following expansion with smooth coefficients,

$$(1.5) \quad u(x, y) \sim \sum_{j=0}^{+\infty} (u_{0j}(y)x^{\frac{n-1}{2}+i\beta+j} + u_{1j}(y)x^{\frac{n-1}{2}-i\beta+j} + u_{2j}(y)x^{n+j}),$$

in the sense that

$$u(x, y) - \sum_{j=0}^k (u_{0j}(y)x^{\frac{n-1}{2}+i\beta+j} + u_{1j}(y)x^{\frac{n-1}{2}-i\beta+j}) = o(x^{\frac{n-1}{2}+k}),$$

with $\beta = \frac{\sqrt{n^2+2n-9}}{2}$ for each $k \geq 0$, if $v = P_1 u$ has an expansion of this form with smooth coefficients and $1 \leq \nu < \frac{n-1}{2}$.

For kernel elements having an expansion with smooth coefficients, one can prescribe the leading terms for them, see Remark 2.2.

Remark 1.1. The ODE result in [9] only gives existence of radially symmetric constant Q -curvature metrics in the conformal class of the hyperbolic metric, but allows the metric to be far away from the hyperbolic metric. As a perturbation result, our theorem gives the existence of solutions in the conformal class of metrics in a small neighborhood of the hyperbolic metric, more precisely, see Theorem 4.1.

Using boundary regularity results and the unique continuation property on the boundary, as a slight extension of the above theorem we have the following result. Note that both boundary regularity results and the unique continuation property approach need x and $h = x^2 g$ to be smooth enough on \overline{M} .

Theorem 1.3. Let (M^n, g) , $n \geq 4$, be a Poincaré-Einstein manifold with the defining function x and the metric $h = x^2 g$ smooth up to the boundary. Suppose also that $L : x^\nu \Lambda^{4,\alpha}(M) \rightarrow x^\nu \Lambda^{0,\alpha}(M)$, where $0 < \nu < \frac{n-1}{2}$ and $0 < \alpha < 1$, is defined in (1.3). Then,

- i) Kernel of L in the weighted space $x^\nu \Lambda^{4,\alpha}(M)$ is of infinite dimension. Also, L is surjective. For each element v in the kernel with its norm small enough, and a given function $Q_{\tilde{g}} \in \Lambda^{0,\alpha}(M, \sqrt{dx dy})$ so that $(Q_{\tilde{g}} - Q_g)$ is in $x^\nu \Lambda^{0,\alpha}(M, \sqrt{dx dy})$ with the norm $\|Q_{\tilde{g}} - Q_g\|_{x^\nu \Lambda^{0,\alpha}}$ small enough, there exists a unique solution u to

the problem (1.2), so that the projection P_1 (see in Theorem 1.5) of u onto $\ker(L)$ is given by v .

ii) Moreover, if $Q_{\tilde{g}} = Q_g$, then u has the expansion result as in Theorem 1.2.

Since this is a perturbation result, we first discuss the linear problem. Using Mazzeo's approach in [15], we obtain the semi-Fredholm property for the linear operator (1.3):

Theorem 1.4. *Let (M^n, g) be an asymptotically hyperbolic manifold with defining function x and the metric $h = x^2 g$ smooth up to the boundary, then the linear operator $L : x^\delta H_e^4(M) \rightarrow x^\delta L^2(M, \sqrt{dx dy})$ as in (1.3), is essentially injective if $\delta > \frac{n}{2}$ and $\delta \neq n + \frac{1}{2}$, with infinite dimensional cokernel, and L is essentially surjective if $\delta < \frac{n}{2}$ and $\delta \neq -\frac{1}{2}$, with infinite dimensional kernel. (Here essentially injective means that the null space of L is at most finitely dimensional, and essentially surjective means that L has closed range and with at most finitely dimensional cokernel.) Moreover, in both cases, L has closed range, and admits a generalized inverse G and orthogonal projectors P_1 onto the nullspace and P_2 onto orthogonal complement of the range of L which are edge operators, such that,*

$$\begin{aligned} GL &= I - P_1, \\ LG &= I - P_2. \end{aligned}$$

The corresponding theorem for the weighted Hölder space is as follows.

Theorem 1.5. *Let (M^n, g) be an asymptotically hyperbolic manifold with defining function x and the metric $h = x^2 g$ smooth up to the boundary. Let $0 < \alpha < 1$. The linear operator $L : x^\nu \Lambda^{4, \alpha}(M) \rightarrow x^\nu \Lambda^{0, \alpha}(M)$ as in (1.3), is essentially injective if $\nu > \frac{n-1}{2}$ and $\nu \neq n$, with infinite dimensional cokernel; and L is essentially surjective if $\nu < \frac{n-1}{2}$ and $\nu \neq -1$, with infinite dimensional kernel. Moreover, in both cases, L has closed range. Also, $x^\nu \Lambda^{4, \alpha}(M)$ has the topological splitting of the following direct sum $x^\nu \Lambda^{4, \alpha}(M) = P_1(x^\nu \Lambda^{4, \alpha}(M)) \oplus (I - P_1)(x^\nu \Lambda^{4, \alpha}(M))$, which are the projection to the null space of L and its topological complement for the second case. Similarly as the theorem with weighted Sobolev spaces, there is a corresponding splitting of $x^\nu \Lambda^{0, \alpha}(M)$ for $\nu > \frac{n-1}{2}$.*

The paper is organized as follows. In Section 2, we study the linear elliptic edge operator L defined in (1.3), and obtain the semi-Fredholm property of the linear operator L . In Section 3, we obtain that if the linear operator L with respect to the initial asymptotically hyperbolic metric g is surjective in a suitable weighted Hölder space, there are infinitely many solutions to the prescribed Q -curvature problem with $Q_{\tilde{g}}$ a small perturbation of Q_g , and the solutions are parametrized by the elements in the kernel of L . Then we give the proof of Theorem 1.2 and Theorem 1.3. Using a special weighted Hölder space, in Section 4, we prove a perturbation result for the prescribed constant Q -curvature problem for a Poincaré-Einstein metric. In Section 5, we give a similar discussion to the prescribed U -curvature equations.

2. SEMI-FREDHOLM PROPERTIES OF THE LINEARIZED OPERATOR

In the following, we will discuss the local parametrix for L and the Fredholm property of L . A clear feature is that the elliptic operators L under consideration here are degenerate near infinity. Here we review some of the material developed by Mazzeo and others in the theory of elliptic edge operators.

As in the introduction, let (M^n, g) be an asymptotically hyperbolic manifold of dimension n , with defining function x and the metric $h = x^2 g$ smooth up to the boundary. Let (x, y) be the local coordinates of M near the boundary, and \mathcal{V}_e be defined in the introduction. The one-forms dual to the vector fields which are elements in \mathcal{V}_e are smooth one forms in M , restricted on the neighborhood of ∂M generated linearly by $\{\frac{dx}{x}, \frac{dy^1}{x}, \dots, \frac{dy^{n-1}}{x}\}$ with coefficients smooth up to ∂M . Generally, a left or right parametrix E of an elliptic operator L on M is a pseudo-differential operator with the property that

$$EL = \text{Id} + R_1, \text{ or } LE = \text{Id} + R_2,$$

with R_1, R_2 compact operators.

The Schwartz kernel of an interior parametrix of the linear operator L is a distribution on $M \times M$, and for "interior" we mean that the parametrix has singularity near the boundary which will be explained in the following. Let (x, y) and (\tilde{x}, \tilde{y}) be local coordinates on each copy of M near the boundary. We know that the parametrix is smooth, except for the singularity along the diagonal $\Delta = \{x = \tilde{x}, y = \tilde{y}\}$, as in the case of compact manifolds. Moreover, here due to the degeneration of the edge operator L , as $x, \tilde{x} \rightarrow 0$, we also have the important additional singularity at the intersection of Δ and the corner, which is $S = \{x = \tilde{x} = 0, y = \tilde{y}\}$. To deal with the boundary singularity, we introduce a new manifold $M_0^2 = M \times_0 M$, by blowing-up $M \times M$ along S . Actually, if we use polar coordinates for $M \times M$ near the corner,

$$\begin{aligned} r &= (x^2 + |y - \tilde{y}|^2 + \tilde{x}^2)^{1/2} \in \mathbb{R}^+, \\ \Theta &= (x, y - \tilde{y}, \tilde{x})/r \in S_{++}^n = \{\Theta \in S^n, \Theta_0, \Theta_n \geq 0\}, \end{aligned}$$

we know that the level set of $r = R$ is a submanifold of dimensional $2n - 1$ for $R > 0$, while $S = \{r = 0\}$ is singular. More precisely, let M_0^2 be the lift of $M \times M$ such that it is the same as $M \times M$ away from S , but near the corner, it is represented by the lift of the polar coordinates, smoothly. Hence, $S_{11} = \{r = 0\}$ is a $(2n - 1)$ -dimensional submanifold of M_0^2 . Let b be the natural projection map from M_0^2 to $M \times M$. For the convenience of calculation, as in [15], we introduce two systems of local coordinates on M_e^2 , $(s, v, \tilde{x}, \tilde{y})$ and (x, y, t, w) , where

$$s = x/\tilde{x}, v = \frac{y - \tilde{y}}{\tilde{x}}; t = \tilde{x}/x, w = \frac{\tilde{y} - y}{x}.$$

Changing variables in these two coordinates,

$$x\partial_x = s\partial_s = x\partial_x - w\partial_w - t\partial_t, \text{ and } x\partial_y = s\partial_v = x\partial_y - \partial_w.$$

In the following with out loss of generality we only need to consider $(s, v, \tilde{x}, \tilde{y})$. Viewing elements in \mathcal{V}_e as first order differential operators, we denote $\text{Diff}_e^*(M)$ the algebra

generated by \mathcal{V}_e with coefficients in the ring $C^\infty(\overline{M})$, and with the product given by composition of operators. Let $\text{Diff}_e^m(M)$ be the linear subspace of differential operators which are of m -th order. Then for $L \in \text{Diff}_e^m(M)$, it has the form

$$(2.1) \quad L = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(x, y) (x\partial_x)^j (x\partial_y)^\alpha,$$

with $a_{j,\alpha} \in C^\infty(\overline{M})$, in the coordinate chart (x, y) . The symbol of L is

$$\sigma_e(L)(x, y; \xi, \eta) = \sum_{j+|\alpha| = m} a_{j,\alpha}(x, y) \xi^j \eta^\alpha.$$

L is elliptic if $\sigma_e(L)(x, y; \xi, \eta) \neq 0$, for $(\xi, \eta) \neq 0$. It is easy to check that Δ_g and the linear operator L in (1.3) are elliptic. L in (2.1) can be considered as a lift to M_e^2 as follows,

$$L = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(x, y) (x\partial_x)^j (x\partial_y)^\alpha = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(s\tilde{x}, \tilde{y} + \tilde{x}v) (s\partial_s)^j (s\partial_v)^\alpha.$$

Let $N(L)$ be the normal operator of L , so that

$$N(L) = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(0, \tilde{y}) (s\partial_s)^j (s\partial_v)^\alpha,$$

is the restriction to S_{11} of the lift of L to M_e^2 . The normal operator is an important approximation of L near the boundary. For the linear operator L in (2.1),

$$L\phi = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(0, y) (x\partial_x)^j (x\partial_y)^\alpha \phi + E\phi,$$

any smooth function ϕ , with the error term

$$E\phi = x \sum_{j+|\alpha| \leq m} b_{j,\alpha}(x, y) (x\partial_x)^j (x\partial_y)^\alpha \phi,$$

for $x > 0$ small, with the coefficients $b_{j,\alpha}$ smooth up to the boundary.

Definition 2.1. The indicial family $I_\zeta(L)$ of $L \in \text{Diff}_e^*(M)$ is defined to be the family of operators

$$L(x^\zeta (\log(x))^p f(x, y)) = x^\zeta (\log(x))^p I_\zeta(L) f(0, y) + O(x^\zeta (\log(x))^{p-1}),$$

for $f \in C^\infty(M)$, $\zeta \in \mathbb{C}$, $p \in \mathbb{N}_0$.

There exists a unique dilation-invariant operator $I(L)$, which is called the indicial operator, such that

$$I(L)(y, s\partial_s) s^\zeta f(y) = s^\zeta I_\zeta(L) f(y).$$

In local coordinates near the boundary, $I(L) = \sum_{j \leq k} a_{j,0}(0, y) (s\partial_s)^j$.

Definition 2.2. If $L \in \text{Diff}_e^*(M)$ is elliptic, we denote $\text{spec}_b(L)$ as the boundary spectrum of L , which is the set of $\zeta \in \mathbb{C}$, for which $I_\zeta(L) = 0$.

Let (M, g) , x , and h be defined as above. Denote S_x as the level set of x (also denoted as x_0 for convenience), and the coordinates $(y_1, \dots, y_{n-1}) = y$. We now use this point of view to deal with our linearized operator (1.3).

In a neighborhood of ∂M , we have the following,

$$(2.2) \quad \text{Ric}_g = \text{Ric}_h + x^{-1}[(n-2)\text{Hess}_h x + \Delta_h x h] - (n-1)x^{-2}|dx|_h^2 h,$$

and

$$(2.3) \quad R_g = -n(n-1)|dx|_h^2 + (2n-2)x(\Delta_h x) + x^2 R_h,$$

where $|dx|_h = 1$, and

$$(\text{Hess}_h)_{ij}(x) = \nabla_i^h \nabla_j^h(x) = \partial_i \partial_j(x) - \Gamma_{ij}^s \partial_s(x) = -\Gamma_{ij}^0 = \frac{1}{2} \partial_x h_{ij} = B_{ij},$$

with B_{ij} the second fundamental form of S_x , for $i, j > 0$; and $(\text{Hess}_h)_{ij}(x) = 0$ otherwise. Also $\Delta_h x = \text{tr}_h(\text{Hess}_h) = H(h)$, with $H(h)$ the mean curvature of the level set of x in the metric h . Here Γ_{ij}^k is the Christoffel symbol with respect to h . Note that Δ_g in our paper is the trace of Hess_g , with negative eigenvalues:

$$(2.4) \quad \Delta_g u = g^{ij}(\partial_i \partial_j - \Gamma_{ij}^k \partial_k) u$$

$$(2.5) \quad = x^2 \Delta_h u + (2-n)x(\nabla_h x, du)$$

$$(2.6) \quad = (2-n)x \partial_x u + x^2(\partial_x^2 u + \Delta_y u + H(h) \partial_x u),$$

where Δ_y is the Laplacian on the level set S_x of x , in the induced metric $h|_{S_x}$.

Near the boundary, the Q -curvature is

$$\begin{aligned} Q_g &= -\frac{2}{(n-2)^2}(n-1)^2 n + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} n^2(n-1)^2 + O(x) \\ &= \frac{n(n^2 - 4)}{8} + O(x), \end{aligned}$$

for $n \geq 5$, and $Q_g = 3 + O(x)$, for $n = 4$.

In the following of this section we will discuss about the linear operator L in (1.3). Note that

$$\begin{aligned} L\phi &= \Delta_g^2 \phi - \text{div}_g(a_n R_g g - b_n \text{Ric}_g) \nabla_g \phi - 4f\phi \\ &= \Delta_g^2 \phi - a_n R_g \Delta_g \phi + b_n \text{Ric}_{ij}^g \nabla_g^i \nabla_g^j \phi - a_n (\nabla_g R_g, \nabla_g \phi) + b_n \nabla_g^i \text{Ric}_{ij} \nabla_g^j \phi - 4f\phi, \\ &= \Delta_g^2 \phi - a_n R_g \Delta_g \phi + b_n \text{Ric}_{ij}^g \nabla_g^i \nabla_g^j \phi + (-a_n + \frac{b_n}{2})(\nabla_g R_g, \nabla_g \phi) - 4f\phi \\ &= \Delta_g^2 \phi - a_n R_g \Delta_g \phi + b_n \text{Ric}_{ij}^g \nabla_g^i \nabla_g^j \phi + \frac{6-n}{2(n-1)} (\nabla_g R_g, \nabla_g \phi) - 4f\phi, \end{aligned}$$

with $f = Q_g$ for $n \geq 5$, and $f = 2Q_g$ for $n = 4$. For the third equality, we use the second Bianchi identity. Also,

$$\Delta_g \phi = x^2 \Delta_h \phi - (n-2)x(\nabla_h x, d\phi)_h = x^2 \Delta_h \phi - (n-2)x \partial_x \phi,$$

and

$$\begin{aligned} R_{ij}(g) \nabla_g^i \nabla_g^j \phi &\sim [-(n-1)x^2 h_{ij} + O(x^3)] x^{-4} \nabla^i \nabla^j \phi \\ &= -(n-1)(\Delta_g \phi + O(x) p(x, y, x\partial_x, x\partial_y)\phi), \end{aligned}$$

for some smooth function $p(\cdot)$. As a consequence,

$$\begin{aligned} L\phi &= \Delta_g^2 \phi - a_n R_g \Delta_g \phi + b_n \text{Ric}_{ij}^g \nabla_g^i \nabla_g^j \phi + \frac{6-n}{2(n-1)} (\nabla_g R_g, \nabla_g \phi) - 4f\phi \\ &= \Delta_g^2 \phi - a_n (-n(n-1) + O(x)) \Delta_g \phi + b_n (-(n-1) \Delta_g \phi \\ &\quad + O(x) p(x, y, x\partial_x, x\partial_y)\phi) + \frac{6-n}{2(n-1)} (-(2n-2)x^2 H(h|_{S_x}) \partial_x \phi \\ &\quad + O(x^3) |\nabla_y \phi|) - \left(\frac{1}{2} n(n^2 - 4) + O(x)\right) \phi, \end{aligned}$$

and then, by definition,

$$N(L) = [(s\partial_s)^2 - (n-1)s\partial_s + s^2\Delta_v - n][(s\partial_s)^2 - (n-1)s\partial_s + s^2\Delta_v + \frac{n^2-4}{2}].$$

In addition,

$$I(L) = ((s\partial_s)^2 - (n-1)s\partial_s - n)((s\partial_s)^2 - (n-1)s\partial_s + \frac{n^2-4}{2}).$$

Let $\phi = s^\zeta$, and $I(L)\phi = 0$. Solving the equation, we get the indicial roots ζ , given by

$$\text{spec}_b(L) = \{n, -1, \frac{n-1}{2} - i \frac{\sqrt{n^2 + 2n - 9}}{2}, \frac{n-1}{2} + i \frac{\sqrt{n^2 + 2n - 9}}{2}\}.$$

Let Λ be the indices set

$$(2.7) \quad \Lambda = \left\{ \frac{1}{2} + \text{Re}(\delta); \delta \in \text{spec}_b(L) \right\}.$$

The operator $N(L)$ acts on functions defined on $\mathbb{R}_s^+ \times \mathbb{R}_v^{n-1}$ for each fixed \tilde{y} , with coordinates (s, v) . For our linear operator L , $N(L)$ does not depend on \tilde{y} . We now take the Fourier transformation of $N(L)$ in v direction,

$$\widehat{N(L)} = \sum_{j+|\alpha| \leq m} a_{i,\alpha} (s\partial_s)^j (i s \eta)^\alpha.$$

We have the symmetry of dilation:

$$a_{j,\alpha} (s\partial_s)^j (s\partial_y)^\alpha = a_{j,\alpha} (ks\partial_{ks})^j (ks\partial_{ky})^\alpha,$$

for any $k \in \mathbb{R} - \{0\}$. Let $t = s|\eta|$, then

$$\widehat{N(L)}(s, \eta) = \sum_{j+|\alpha| \leq m} a_{i,\alpha}(0, \tilde{y})(t\partial_t)^j (i t \hat{\eta})^\alpha,$$

which is denoted as $L_0(t, \hat{\eta})$, where $\hat{\eta} = \frac{\eta}{|\eta|}$. This is a family of totally characteristic operators on \mathbb{R}_+^n and generally its coefficients depend on \tilde{y} . Now we have fixed $\hat{\eta}$ in the formula, and it has no scaling freedom in this direction.

Let $\mathcal{H}^{m,\delta,l}$ be the weighted Sobolev space

$$\mathcal{H}^{m,\delta,l} = \{f : \phi(t)f \in t^\delta H_e^m(\mathbb{R}^+), (1 - \phi(t))f \in t^{-l} H^m(\mathbb{R}^+)\},$$

with $\phi \in C_0^\infty(\mathbb{R}^+)$, and $\phi(t) = 1$ in a neighborhood of $t = 0$. Note that

$$L_0 : t^\delta \mathcal{H}^{m,\delta,l} \rightarrow t^\delta \mathcal{H}^{m-4,\delta,l+4}$$

is bounded.

For our linear operator L ,

$$\widehat{N(L)} = [(s\partial_s)^2 - (n-1)s\partial_s + s^2(-|\eta|^2) - n][(s\partial_s)^2 - (n-1)s\partial_s + s^2(-|\eta|^2) + \frac{n^2-4}{2}],$$

and then

$$\begin{aligned} L_0(t, \widehat{\eta}) &= [(t\partial_t)^2 - (n-1)t\partial_t - t^2 - n][(t\partial_t)^2 - (n-1)t\partial_t - t^2 + \frac{n^2-4}{2}] \\ &= L_1 \circ L_2, \end{aligned}$$

with $s\partial_s = s|\eta| \partial_{s|\eta|} = t\partial_t$, and L_0 here does not depend on \tilde{y} . Now we have used the full symmetry of the operator, and made it into the simplest form.

Let us consider the relationship of Fredholm property among $N(L)$, $\widehat{N(L)}$ and L_0 , in $t^\delta L^2$, for $\delta > \frac{n}{2}$. We know that the first two operators have the same properties of injectivity and surjectivity. Let

$$L_0\varphi(t) = 0,$$

by definition, it holds if and only if

$$\widehat{N(L)}\varphi(s|\eta|) = 0.$$

But then

$$\widehat{N(L)}(a(\eta)\varphi(s|\eta|)) = a(\eta)\widehat{N(L)}\varphi(s|\eta|) = 0,$$

for all $a(\eta)$ smooth, since the derivative is only in s direction, with fixed η . Then, using the inverse Fourier transformation,

$$N(L) \int_{\mathbb{R}^{n-1}} e^{2\pi i \langle y, \eta \rangle} a(\eta) \varphi(s|\eta|) d\eta = 0.$$

This means kernel of one dimensional L_0 corresponds to the infinite dimensional kernel of $N(L)$, and this construction also gives the fact that the kernel of $N(L)$ is either trivial or of infinite dimension. But if $\widehat{N(L)}$ is injective, then L_0 is injective. Conversely, if L_0 is injective, then $\widehat{N(L)}$ is injective, and so is $N(L)$. We have a dual argument of the surjectivity for $\delta < \frac{n}{2}$. As in [15], L_0 is Fredholm when $\delta \notin \Lambda$, with the set Λ in (2.7), and $N(L)$ is semi-Fredholm with either infinite dimensional kernel or cokernel. Roughly speaking, L is a small perturbation of $N(L)$ near ∂M . When $N(L)$ is injective or surjective, L is essentially injective or essentially surjective, which will be Theorem 1.4 and Theorem 1.5.

To see the semi-Fredholm property of L , the strategy is to first study the Fredholm property of L_0 and $N(L)$, and finally obtain the semi-Fredholm property of L using Mazzeo's theorems which we list here as Theorem 2.4 and Corollary 2.5.

Now we discuss on the Fredholm property of L_0 , L_1 and L_2 on the weighted spaces. To this end, we introduce Bessel functions as solutions to the Bessel equation as follows, which is well studied,

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \alpha^2)y = 0,$$

where α is a complex number.

The Bessel functions I_α and $I_{-\alpha}$ form a basis of linear space of solutions to the Bessel function above, while $\{I_\alpha, K_\alpha\}$ is another basis. For $\text{Re}(\alpha) > -\frac{1}{2}$, and $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$, the integral representations of these solutions are as follows,

$$I_\alpha(x) = \frac{\left(\frac{x}{2}\right)^\alpha}{\Gamma(\alpha + \frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^1 e^{-xt}(1 - t^2)^{\alpha - \frac{1}{2}} dt,$$

$$K_\alpha(x) = \frac{\pi}{2} \frac{I_\alpha(x) - I_{-\alpha}(x)}{\sin(\alpha\pi)} = \frac{\Gamma(\frac{1}{2})\left(\frac{x}{2}\right)^\alpha}{\Gamma(\alpha + \frac{1}{2})} \int_1^\infty e^{-xt}(t^2 - 1)^{\alpha - \frac{1}{2}} dt,$$

with x a complex number. See Page 172 and 77 in [22]. Note that I_α is bounded near $x = 0$, and it increases exponentially near $+\infty$, and

$$K_\alpha(x) \sim C(\varepsilon)x^{\text{Re}(\alpha)}e^{-x+\varepsilon},$$

for any $\varepsilon > 0$, as $x \rightarrow +\infty$. Also $K_\alpha(x)$ is bounded for $\text{Re}(\alpha) \geq 0$, near $x = 0$. The form $K_\alpha(x)$ is more useful near $x = \infty$, since it decays exponentially.

We want to solve the following ODE, by transferring it into the Bessel type equations as above.

$$L_1 u = ((t\partial_t)^2 - (n-1)t\partial_t - t^2 - n)u = 0.$$

Let $u = t^\beta \tilde{u}$, then we obtain that

$$(2.8) \quad t^\beta((t\partial_t)^2 \tilde{u} + (2\beta + 1 - n)t\partial_t \tilde{u} + (-n - t^2 + \beta^2 - \beta(n-1))\tilde{u}) = 0.$$

Then, letting $2\beta + 1 - n = 0$, the equation (2.8) is just the form of the Bessel function defined as above. In this case, $\beta = \frac{n-1}{2}$, and then the index $\alpha = \frac{n+1}{2}$. Therefore,

$$u(t) = t^{\frac{n-1}{2}}(C_1 I_{\frac{n+1}{2}}(t) + C_2 K_{\frac{n+1}{2}}(t)).$$

In fact,

$$t^{\frac{n-1}{2}} I_{\frac{n+1}{2}}(t|\eta|) \sim t^n |\eta|^{\frac{n+1}{2}}, \quad t^{\frac{n-1}{2}} K_{\frac{n+1}{2}}(t|\eta|) \sim t^{-1} |\eta|^{-\frac{n+1}{2}},$$

near $t = 0$. Moreover,

$$t^{\frac{n-1}{2}} I_{\frac{n+1}{2}}(t|\eta|) \sim t^{\frac{n}{2}-1} e^{t|\eta|} / \sqrt{2\pi|\eta|}, \quad t^{\frac{n-1}{2}} K_{\frac{n+1}{2}}(t|\eta|) \sim t^{\frac{n}{2}-1} e^{-t|\eta|} \sqrt{\frac{\pi}{2|\eta|}},$$

as $t \rightarrow \infty$.

Similarly,

$$L_2 u = ((t\partial_t)^2 - (n-1)t\partial_t - t^2 + \frac{n^2 - 4}{2})u = 0.$$

Let $u(t) = t^\beta \tilde{u}(t)$, then

$$t^\beta ((t\partial_t)^2 \tilde{u} + (2\beta + 1 - n)t\partial_t \tilde{u} + (\frac{n^2 - 4}{2} - t^2 + \beta^2 - \beta(n - 1))\tilde{u}) = 0.$$

Set $2\beta + 1 - n = 0$, so that $\beta = \frac{n-1}{2}$, and then \tilde{u} is a solution to the Bessel equation with $\alpha = \frac{i\sqrt{n^2+2n-9}}{2}$.

$$u(t) = t^{\frac{n-1}{2}} (C_1 I_{\frac{i\sqrt{n^2+2n-9}}{2}}(t) + C_2 K_{\frac{i\sqrt{n^2+2n-9}}{2}}(t)).$$

By the expansion of the series form of the Bessel functions, as in [[13], P. 108], we have

$$t^{\frac{n-1}{2}} I_\alpha(t|\eta|) \sim t^{\frac{n-1}{2}+\alpha} |\eta|^\alpha / (2^\alpha \Gamma(1 + \alpha)),$$

and

$$t^{\frac{n-1}{2}} I_{-\alpha}(t|\eta|) \sim t^{\frac{n-1}{2}-\alpha} |\eta|^\alpha / (2^\alpha \Gamma(1 - \alpha)),$$

with $\alpha = \frac{i\sqrt{n^2+2n-9}}{2}$, near $t = 0$. Now it is easy to see that the linear combination

$$x^{\frac{n-1}{2}} (C_1 x^{i\frac{\sqrt{n^2+2n-9}}{2}} + C_2 x^{-i\frac{\sqrt{n^2+2n-9}}{2}})$$

can never vanish to infinite order at $t = 0$ if either $C_1 \neq 0$ or $C_2 \neq 0$. Also,

$$t^{\frac{n-1}{2}} K_\alpha(t|\eta|) \sim t^{\frac{n-1}{2}} \frac{\pi}{2} \frac{I_\alpha(t|\eta|) - I_{-\alpha}(t|\eta|)}{\sin(\alpha\pi)},$$

with $\alpha = \frac{i\sqrt{n^2+2n-9}}{2}$, and $|\eta| \neq 0$, near $t = 0$.

Using the integral form as above, we have that $I_\alpha(t)$ grows exponentially, while K_α decays exponentially as $t \rightarrow +\infty$, for $\alpha = \frac{i\sqrt{n^2+2n-9}}{2}$.

Denote L_0^t to be the L^2 adjoint of L_0 in the measure dt , and

$$L_0^* = t^{2\delta} L_0 t^{-2\delta},$$

to be the adjoint of L_0 in $t^\delta L^2$ in the measure $t^{-2\delta} dt$. These are all elliptic operators, with boundary spectra:

$$\begin{aligned} \text{spec}_b(L_0^t) &= \{-\zeta - 1 : \zeta \in \text{spec}_b(L_0)\}, \\ \text{spec}_b(L_0^*) &= \{-\zeta + 2\delta - 1 : \zeta \in \text{spec}_b(L_0)\}. \end{aligned}$$

For example, for $L_1 = (t\partial_t)^2 - (n-1)(t\partial_t) - t^2 - n$,

$$\int L_1 u v dt = \int u L_1^t v dt.$$

Then

$$L_1^t = (-\partial_t(t\cdot))^2 + (n-1)(\partial_t(t\cdot)) - t^2 - n,$$

with

$$\partial_t(t\cdot) = t\partial_t + 1,$$

and $p^t(\xi) = p(-(\xi + 1))$, for the quadratic polynomial p . Also, for L_1^* , using the fact that

$$-\partial_t(t t^{-2\delta} \cdot) = -t^{-2\delta}(-2\delta + 1 + t \partial_t) = t^{-2\delta}(2\delta - 1 - t \partial_t),$$

and

$$\int L_1 u v t^{2\delta} dt = - \int u t^{-2\delta} L_1^t(t^{2\delta} v) t^{2\delta} dt,$$

we obtain the boundary spectra as listed above. For the fourth order differential equation, we have obtained four linearly independent solutions, and they generalize the solution space.

Let $\delta = \frac{n-1}{2} + \frac{1}{2} = \frac{n}{2}$, we have $L_1^* = L_1$, and $L_2^* = L_2$.

Definition 2.3. *We say that an operator L has the unique continuation property on a boundary B if any solution of $Lu = 0$ vanishing to infinite order at B vanishes identically.*

Hypothesis 1. *For each \tilde{y} and $\hat{\eta}$, both L_0 and its adjoint L_0^* (The dual of L_0 with respect to the space $t^{\text{Re}\delta} L^2$ for any δ we need) have the unique continuation property at $\{t = 0\}$.*

We know from the discussion above that L_0 satisfies the unique continuation property. Under the continuation hypothesis, we have that for each element $(\tilde{y}, \hat{\eta}) \in N_0$, L_0 is surjective on $x^\delta L^2$ or injective on $x^\delta L^2$ when δ is sufficiently negative or sufficiently large. For our case, we use $\delta = \frac{n}{2}$ in Hypothesis 1. Now let us define $\bar{\delta}$ to be the minimal value of δ so that L_0 is injective, and meanwhile $\underline{\delta}$ the maximal value so that L_0 is surjective dually. These values must lie in Λ . The following theorem and corollary tell us the relationship between semi-Fredholm properties of L and the Fredholm properties of L_0 , for certain cases we need.

Theorem 2.4. *(Theorem 6.1. in [15]) Suppose $L \in \text{Diff}_e^n(M)$ is elliptic and satisfies the unique continuation hypothesis, and that $\text{spec}_b(L)$ is discrete. Suppose also that $\delta \notin \Lambda$ is chosen so that either $\delta > \bar{\delta}$ or $\delta < \underline{\delta}$. Then $L : x^\delta H_e^{r+m}(M) \rightarrow x^\delta H_e^r(M)$ has closed range, and it is either essentially surjective, or essentially injective, which means respectively that L has either an at most finite dimensional nullspace, or a finite dimensional cokernel. Therefore, it admits a generalized inverse G and orthogonal projectors P_i onto the nullspace and orthogonal complement of the range of L which are edge operators, such that,*

$$GL = I - P_1,$$

$$LG = I - P_2.$$

Since the edge operators used in the proof of the weighted Sobolev spaces are bounded in the appropriate Hölder spaces, the corresponding result for Hölder spaces follows.

Corollary 2.5. *(Corollary 6.4. in [15]) For L as in Theorem 2.4, $k \geq m$ a positive integer and $0 < \alpha < 1$ the mapping $L : x^\nu \Lambda^{k,\alpha} \rightarrow x^\nu \Lambda^{k-m,\alpha}$ is semi-Fredholm provided $\nu = \delta - \frac{1}{2}$ and $\delta \notin \Lambda$ is as in the previous theorem. If $\delta < \underline{\delta}$ or $\delta > \bar{\delta}$ so that L is essentially surjective or essentially injective, then topologically, we have the splitting,*

$$x^\nu \Lambda^{k,\alpha} = P_1(x^\nu \Lambda^{k,\alpha}) \oplus (I - P_1)(x^\nu \Lambda^{k,\alpha}),$$

$$x^\nu \Lambda^{k-m,\alpha} = P_2(x^\nu \Lambda^{k-m,\alpha}) \oplus (I - P_2)(x^\nu \Lambda^{k-m,\alpha}).$$

Let us compute $\bar{\delta}$ and $\underline{\delta}$ for L_0 . First, for L_1 , since $t^{\frac{n-1}{2}} I_{\frac{n+1}{2}}(t|\eta|)$ increases exponentially as t goes to ∞ (here $|\eta| \neq 0$), it does not lie in $t^\delta L^2$ for any $\delta > 0$; furthermore,

$$t^{\frac{n-1}{2}} K_{\frac{n+1}{2}}(t|\eta|) \in t^\delta L^2(\mathbb{R}_+),$$

for $\delta < -\frac{1}{2}$. Similarly, for L_2 , $t^{\frac{n-1}{2}} I_{\frac{i\sqrt{n^2+2n-9}}{2}}(t|\eta|)$ grows exponentially when t goes to ∞ (with $|\eta| \neq 0$), and

$$t^{\frac{n-1}{2}} K_{\frac{i\sqrt{n^2+2n-9}}{2}}(t|\eta|) \in t^\delta L^2(\mathbb{R}_+),$$

for $\delta < \frac{n-1}{2} + \frac{1}{2} = \frac{n}{2}$. Therefore, L_1 and L_2 both have trivial kernel in the space $x^\delta L^2(M, \sqrt{dx dy})$ for $\delta > \frac{n}{2}$. But $\text{Ker}(L_2)$ is nontrivial for $\delta < \frac{n}{2}$. Also the composition of two injective map is still injective. Therefore, $\bar{\delta} = \frac{n}{2}$ for $L_0 = L_1 \circ L_2$. Since L_0 is self-adjoint in $t^{\frac{n}{2}} L^2(\mathbb{R}_+)$, we have that $\underline{\delta} = \frac{n}{2}$. Since it satisfies the conditions of Theorem 2.4 and Corollary 2.5, therefore Theorems 1.4 and 1.5 are proved. \square

To conclude this section, we want to see when L is injective or surjective in the special case of Poincaré-Einstein manifolds. For a Poincaré-Einstein manifold (M, g) with $g = x^{-2}h$, without loss of generality we assume $R_g = -n(n-1)$. Let us first consider it in the weighted Sobolev spaces. We have

$$L = (\Delta_g - n)(\Delta_g + \frac{(n+2)(n-2)}{2}) = \mathcal{T}_1 \circ \mathcal{T}_2.$$

We know that L is self-adjoint with respect to $x^{\frac{n}{2}} L^2(M, \sqrt{dx dy})$. Then to show that $L : x^\delta H_e^4(M) \rightarrow x^\delta L^2(M)$ is surjective for $0 \leq \delta < \frac{n}{2}$, we only need to show that L is injective when $\delta > \frac{n}{2}$. For that, we only need to show that \mathcal{T}_1 and \mathcal{T}_2 are injective for $\delta > \frac{n}{2}$.

As a special case, if \mathcal{T}_1 and \mathcal{T}_2 are injective in $L^2(M, g)$, which is $x^{\frac{n}{2}} L^2(M, \sqrt{dx dy})$, then we are done. For the Poincaré ball (B, g_{-1}) , we know that the Laplacian $-\Delta_g$ has pure continuous spectrum, consisting of $[\frac{(n-1)^2}{4}, \infty)$, with $\lambda_0 = \frac{(n-1)^2}{4}$. So \mathcal{T}_1 is injective when $\delta > \frac{n}{2}$ in this special case.

As mentioned in the introduction, the way we prove the surjectivity of \mathcal{T}_2 in the following involves unique continuation property and boundary smoothness argument, which need x and h to be smooth enough. So we assume the defining function x and the metric h to be smooth up to the boundary.

Lemma 2.6. $\mathcal{T}_1, \mathcal{T}_2 : x^\delta H_e^{2+m}(M, \sqrt{dx dy}) \rightarrow x^\delta H_e^m(M, \sqrt{dx dy})$, are both injective for $\delta > \frac{n}{2}$, and all $m \geq 0$.

Proof of Lemma 2.6. By the regularity argument, we only need to discuss on the case $m = 0$. The proof is as follows.

If $u \in x^\delta L^2(M, \sqrt{dx dy})$, for $\delta > \frac{n}{2}$, then $u \in L^2(M, g)$. Moreover, if also

$$\mathcal{T}_1 u = (\Delta_g - n)u = 0,$$

by Weyl's lemma, $u \in H^1(M, g)$. Now we multiply u on both sides of the equation, and integrate by parts, and then we have

$$-\int_M (|\nabla u|_g^2 + u^2) dV_g = 0.$$

Therefore, $u = 0$. Then we have that \mathcal{T}_1 is injective.

Let us consider the equation

$$(2.9) \quad \mathcal{T}_2 u = (\Delta_g + \frac{n^2 - 4}{2}) u = 0.$$

First, we know from [18] that $(-\Delta_g - \lambda)$ satisfies the unique continuation property, for any constant real number λ (See Corollary 11 in [18]), namely, if $u \in C^\infty(M)$ satisfies $(-\Delta_g - \lambda)u = 0$, and u vanishes to infinite order along an open set of ∂M , then $u = 0$. Moreover, in [18], combining the boundary regularity result and the unique continuation result for $(-\Delta_g - \lambda)$, it was proved in [18] that if $\lambda > \frac{(n+1)^2}{4}$, $u \in L^2(M, g)$ and $(-\Delta_g - \lambda)u = 0$ then $u = 0$. It is easy to check that when $n \geq 5$, $\frac{n^2-4}{2} > \frac{(n+1)^2}{4}$. Therefore, for $n \geq 5$, \mathcal{T}_2 is injective in $L^2(M, g) = x^{\frac{n}{2}}L^2(M, \sqrt{dxdy})$.

When $n = 4$, since $\frac{n^2-4}{2} < \frac{(n+1)^2}{4}$, we can not use his result directly. But since we still have the unique continuation property for \mathcal{T}_2 , we only need to prove the boundary regularity of u . Actually, for our case we do not require $u \in L^2(M, g)$ but allow $u \in x^\delta L^2(M, \sqrt{dxdy})$, for $\delta > \frac{n}{2}$, and the method of proving the boundary regularity still works here. For completeness, we give the details here. Assume $u \in x^\delta L^2(M, \sqrt{dxdy})$ for $\delta > \frac{n}{2}$ satisfies equation (2.9). Let us denote the indicial roots of \mathcal{T}_2 as s_1, s_2 . Using the boundary expansion from Section 7 in [15], we have that

$$(2.10) \quad u \sim \sum_{j=0}^{\infty} \left(\sum_{p=0}^{N_1} x^{s_1+j} (\log(x))^p u_{1,j,p}(y) + \sum_{p=0}^{N_2} x^{s_2+j} (\log(x))^p u_{2,j,p}(y) \right).$$

Since the real parts of the indicial roots s_1, s_2 are both $\frac{n-1}{2}$, which is less than $\delta - \frac{1}{2}$, then by Theorem (7.17) in [15], we have that $u_{1,0,p}$ and $u_{2,0,p}$ vanish for all p , and that the coefficients $u_{i,j,p}(y)$ are all smooth, and by Theorem (7.3) in [15], using the substitution of the expansion into the equation, we have that the coefficients all vanish by induction. Then, by the unique continuation property, we have that $u = 0$.

It follows that \mathcal{T}_1 and \mathcal{T}_2 are both injective when $\delta > \frac{n}{2}$. This proves the lemma. \square

The lemma implies that L is injective for $\delta > \frac{n}{2}$ on the Poincaré-Einstein manifolds. Since L is self-adjoint in $x^{\frac{n}{2}}L^2(M, \sqrt{dxdy})$, then L is surjective when $0 < \delta < \frac{n}{2}$.

The linear edge operators used above are all bounded linear operators in the weighted Hölder spaces, and can be used correspondingly in the weighted Hölder spaces. Then the corresponding statement for the weighted Hölder spaces is as follows. Let

$$L : x^\nu \Lambda^{4,\alpha}(M) \rightarrow x^\nu \Lambda^{0,\alpha}(M).$$

Here $0 < \alpha < 1$. Then L is injective when $\nu = \delta - \frac{1}{2} > \frac{n-1}{2}$, while L is surjective when $0 < \nu = \delta - \frac{1}{2} < \frac{n-1}{2}$ on the Poincaré-Einstein manifolds M .

Remark 2.1. Generally, on an asymptotically hyperbolic manifold (M, g) with a smooth defining function x and $h = x^2g$ smooth, let $u \in \text{Ker}(L)$ for L defined in (1.3) in the

weighted Hölder spaces $x^\nu \Lambda^{4,\alpha}(M, \sqrt{dx dy})$, for $0 < \nu < \frac{n-1}{2}$ and $0 < \alpha < 1$. Then, $u \in x^\nu \Lambda^{m,\alpha}$ for all $m \in \mathbb{N}$, and u has the following weak expansion with coefficients which are generally distributions,

(2.11)

$$u(x, y) \sim \sum_{j=0}^{+\infty} (u_{0j}(y) x^{\frac{n-1}{2} + i\frac{\sqrt{n^2+2n-9}}{2} + j} + u_{1j}(y) x^{\frac{n-1}{2} - i\frac{\sqrt{n^2+2n-9}}{2} + j} + x^{n+j} u_{2j}(y)),$$

in the sense that

$$u(x, y) - \sum_{j=0}^k (u_{0j}(y) x^{\frac{n-1}{2} + i\frac{\sqrt{n^2+2n-9}}{2} + j} + u_{1j}(y) x^{\frac{n-1}{2} - i\frac{\sqrt{n^2+2n-9}}{2} + j}) = o(x^{\frac{n-1}{2} + k}),$$

for $k \geq 0$. If either u_{00} or u_{01} is smooth, then all the coefficients are smooth. The more precise regularity of the coefficients in a weighted Sobolev space setting can be found in Chapter 7 in [15].

Remark 2.2. On a Poincaré-Einstein manifold (M, g) with a smooth defining function x and $h = x^2 g$ smooth, for $0 < \nu < \frac{n-1}{2}$ and $0 < \alpha < 1$, since \mathcal{T}_1 is injective, an element u in the kernel of L is exactly an element in the kernel of \mathcal{T}_2 . By Proposition 3.4. in [8], for any chosen $u_{00} \in C^\infty$ or $u_{10} \in C^\infty$, there exists a unique $u \in x^\nu \Lambda^{4,\alpha}(M, \sqrt{dx dy})$, for $0 < \nu < \frac{n-1}{2}$, in the kernel of L , so that u has the expansion (2.11) with smooth coefficients.

3. THE NONLINEAR PROBLEM

Now let us return to the perturbation problem. It is more convenient to work in weighted Hölder spaces. Let (M, g) be an asymptotically hyperbolic manifold defined as in the introduction. Let \tilde{g} , u also be defined as in the introduction, and let the prescribed curvature $Q_{\tilde{g}} = f$. Define the operator $\mathcal{T} : x^\nu \Lambda^{4,\alpha}(M) \rightarrow x^\nu \Lambda^{0,\alpha}(M)$ as follows,

$$\mathcal{T}(u) = \begin{cases} 2f e^{4u} - 2Q_g - 8Q_g u, & n = 4, \\ \frac{n-4}{2} (1 + u)^{\frac{n+4}{n-4}} f - \frac{n-4}{2} Q_g - \frac{n+4}{2} Q_g u, & n \geq 5. \end{cases}$$

We rewrite it in the form

$$\mathcal{T}(u) = \begin{cases} 2(e^{4u} - 1 - 4u)f + 2(f - Q_g) - 8(Q_g - f)u, & n = 4, \\ \frac{n-4}{2}((1 + u)^{\frac{n+4}{n-4}} - 1 - \frac{n+4}{n-4}u)f + \frac{n-4}{2}(f - Q_g) + \frac{n+4}{2}(f - Q_g)u, & n \geq 5. \end{cases}$$

Let L be as in (1.3), then the prescribed Q -Curvature equation is

$$(3.1) \quad Lu = \mathcal{T}(u).$$

Let $0 < \nu < \underline{\nu} = \frac{n-1}{2}$ and $0 < \alpha < 1$, so that L is essentially surjective. Moreover, in the following we assume that L is surjective. Then

$$L : V_1 = (I - P_1)(x^\nu \Lambda^{4,\alpha}(M)) \rightarrow x^\nu \Lambda^{0,\alpha}(M)$$

is an isomorphism, using topological splitting of $x^\nu \Lambda^{4,\alpha}(M)$ in Theorem 1.5 and the open mapping theorem. That is,

$$(3.2) \quad C_1 \|u\|_{x^\nu \Lambda^{4,\alpha}(M)} \leq \|L u\|_{x^\nu \Lambda^{0,\alpha}(M)} \leq C_2 \|u\|_{x^\nu \Lambda^{4,\alpha}(M)},$$

for some constant $C_2 > C_1 > 0$, for all $u \in V_1$. We denote the inverse of L as

$$L^{-1} : x^\nu \Lambda^{0,\alpha}(M) \rightarrow V_1.$$

Let $f \in C^\alpha(M)$, and

$$(Q_g - f) \in x^\nu \Lambda^{0,\alpha},$$

with its small norm to be determined later. We want to use elements in kernel of L to parametrize the perturbation solutions to the nonlinear problem at 0. We will define a new map for each element in the kernel of L , and use it to construct a contraction map. For any fixed $u_1 \in \text{Ker}(L)$, for any $u_2 \in V_1$, let $u = u_1 + u_2$, and

$$\mathcal{T}_{u_1}(u_2) = \mathcal{T}(u_1 + u_2).$$

Now $L^{-1} \circ \mathcal{T}_{u_1} : V_1 \rightarrow V_1$.

From now on, let u_1 be any fixed element in $B_\epsilon(0) \cap \text{Ker}(L)$, and $u_2 \in B_\epsilon(0) \cap V_1$, with small $\epsilon \in (0, 1)$ to be determined. Note that

$$\|\mathcal{T}_{u_1}(u_2)\|_{x^\nu \Lambda^{0,\alpha}} \leq \begin{cases} 2\|(e^{4u} - 1 - 4u)f\|_{x^\nu \Lambda^{0,\alpha}(M)} + 2\|(f - Q_g)\|_{x^\nu \Lambda^{0,\alpha}(M)} \\ + 8\|(f - Q_g)u\|_{x^\nu \Lambda^{0,\alpha}(M)}, & n = 4, \\ \frac{n-4}{2}\|((1+u)^{\frac{n+4}{n-4}} - 1 - \frac{n+4}{n-4}u)f\|_{x^\nu \Lambda^{0,\alpha}(M)} \\ + \frac{n-4}{2}\|(f - Q_g)\|_{x^\nu \Lambda^{0,\alpha}(M)} + \frac{n+4}{2}\|(f - Q_g)u\|_{x^\nu \Lambda^{0,\alpha}(M)}, & n \geq 5. \end{cases}$$

Then we have

$$\begin{aligned} \|\mathcal{T}_{u_1}(u_2)\|_{x^\nu \Lambda^{0,\alpha}} &\leq C(n)(\|f\|_{L^\infty}\|(u_1 + u_2)^2\|_{x^\nu \Lambda^{0,\alpha}} + (1 + \|u_1 + u_2\|_{L^\infty})\|f - Q_g\|_{x^\nu \Lambda^{0,\alpha}} \\ &\quad + \|x^{-\nu}(u_1 + u_2)\|_{L^\infty}\|(u_1 + u_2)\|_{L^\infty}(\|f\|_{\Lambda^{0,\alpha}} + \|Q_g\|_{\Lambda^{0,\alpha}}) \\ &\quad + \|f - Q_g\|_{L^\infty}\|u_1 + u_2\|_{x^\nu \Lambda^{0,\alpha}}). \end{aligned}$$

where $C > 0$ is a constant depending only on n , the diameter of M and ν . By the definition of the weighted norm,

$$(3.3) \quad \|\phi\|_{L^\infty} \leq \|\phi\|_{\Lambda^{0,\alpha}}, \text{ and } \|\phi\|_{L^\infty} \leq C_0 \|\phi\|_{x^\nu \Lambda^{0,\alpha}},$$

for a constant $C_0 > 0$ depending on the defining function and ν , for any $\phi \in x^\nu \Lambda^{0,\alpha}$. Therefore,

$$\begin{aligned} \|\mathcal{T}_{u_1}(u_2)\|_{x^\nu \Lambda^{0,\alpha}} &\leq C_1 ((\epsilon(\|f\|_{\Lambda^{0,\alpha}} + \|Q_g\|_{\Lambda^{0,\alpha}}) + \|f - Q_g\|_{L^\infty})\|u_1 + u_2\|_{x^\nu \Lambda^{0,\alpha}} \\ &\quad + (1 + \epsilon)\|f - Q_g\|_{x^\nu \Lambda^{0,\alpha}}), \end{aligned}$$

where C_1 depends on n , the defining function, the diameter of M and ν , so that

$$\begin{aligned} \|L^{-1} \circ \mathcal{T}_{u_1}(u_2)\|_{x^\nu \Lambda^{4,\alpha}} &\leq C ((\epsilon(\|f\|_{\Lambda^{0,\alpha}} + \|Q_g\|_{\Lambda^{0,\alpha}}) + \|f - Q_g\|_{L^\infty})\|u_1 + u_2\|_{x^\nu \Lambda^{0,\alpha}} \\ &\quad + (1 + \epsilon)\|f - Q_g\|_{x^\nu \Lambda^{0,\alpha}}), \end{aligned}$$

where $C = C_1 \|L^{-1}\|$ depends on the defining function, the diameter of M , ν , n and $\|L^{-1}\|$. We choose $\epsilon \in (0, 1)$ small so that

$$(3.4) \quad 16 C \epsilon \|Q_g\|_{\Lambda^{0,\alpha}} < 1,$$

and let f satisfy that

$$(3.5) \quad \|f\|_{\Lambda^{0,\alpha}} \leq 2 \|Q_g\|_{\Lambda^{0,\alpha}}, \text{ and } \|f - Q_g\|_{x^\nu \Lambda^{0,\alpha}} \leq \min\left\{\frac{1}{4(1+\epsilon)C}\epsilon, \frac{\epsilon \|Q_g\|_{\Lambda^{0,\alpha}}}{C_0}\right\}.$$

Combining (3.3), we have

$$\|L^{-1} \circ \mathcal{T}_{u_1}(u_2)\|_{x^\nu \Lambda^{4,\alpha}} \leq \frac{3}{4}\epsilon.$$

Therefore, $L^{-1} \circ T_{u_1}$ maps $B_\epsilon(0) \cap V_1$ into $B_\epsilon(0) \cap V_1$.

For $u_3, u_4 \in V_1 \cap B_\epsilon(0)$,

$$\begin{aligned} & \|L^{-1} \circ \mathcal{T}_{u_1}(u_3) - L^{-1} \circ \mathcal{T}_{u_1}(u_4)\|_{x^\nu \Lambda^{4,\alpha}} \\ & \leq \|L^{-1}\| \|\mathcal{T}_{u_1}(u_3) - \mathcal{T}_{u_1}(u_4)\|_{x^\nu \Lambda^{0,\alpha}} \\ & = \begin{cases} \|L^{-1}\| \|2f(e^{4u_1}(e^{4u_3} - e^{4u_4}) - 4(u_3 - u_4)) - 8(Q_g - f)(u_3 - u_4)\|_{x^\nu \Lambda^{0,\alpha}}, & n = 4, \\ \|L^{-1}\| \left\| \frac{n-4}{2}((1+u_1+u_3)^{\frac{n+4}{n-4}} - (1+u_1+u_4)^{\frac{n+4}{n-4}} - \frac{n+4}{n-4}(u_3 - u_4))f \right. \\ \quad \left. + \frac{n+4}{2}(f - Q_g)(u_3 - u_4) \right\|_{x^\nu \Lambda^{0,\alpha}}, & n \geq 5. \end{cases} \end{aligned}$$

But

$$e^{4(u_1+u_3)} - e^{4(u_1+u_4)} - 4(u_3 - u_4) = 4(u_3 - u_4)w,$$

with

$$w = \left(\frac{e^{4(u_1+u_3)} - e^{4(u_1+u_4)}}{4(u_3 - u_4)} - 1 \right) = \left(\int_0^1 e^{4(u_1+u_4+t(u_3-u_4))} dt - 1 \right) \in x^\nu \Lambda^{0,\alpha} \cap B_{C\epsilon}(0),$$

with C which does not depend on u_3, u_4 , or $\epsilon \in (0, 1)$. We have similar results for $n \geq 5$. By the discussion above,

$$\begin{aligned} & \|L^{-1} \circ \mathcal{T}_{u_1}(u_3) - L^{-1} \circ \mathcal{T}_{u_1}(u_4)\|_{x^\nu \Lambda^{4,\alpha}} \\ & \leq \|L^{-1}\| \widetilde{C}_0 (\epsilon \|f\|_{\Lambda^{0,\alpha}} \|u_3 - u_4\|_{x^\nu \Lambda^{0,\alpha}} + \|Q_g - f\|_{x^\nu \Lambda^{0,\alpha}} \|u_3 - u_4\|_{x^\nu \Lambda^{0,\alpha}}) \\ & = \|L^{-1}\| \widetilde{C}_0 (\epsilon \|f\|_{\Lambda^{0,\alpha}} + \|Q_g - f\|_{x^\nu \Lambda^{0,\alpha}}) \|u_3 - u_4\|_{x^\nu \Lambda^{0,\alpha}}, \quad n \geq 4, \end{aligned}$$

where \widetilde{C}_0 depends only on the defining function, the diameter of M , ν and n . Let ϵ be small so that

$$(3.6) \quad 8\widetilde{C}_0 \|L^{-1}\| (1 + \|Q_g\|_{\Lambda^{0,\alpha}}) \epsilon < 1,$$

and let

$$(3.7) \quad \|Q_g - f\|_{x^\nu \Lambda^{0,\alpha}} \leq \frac{1}{8\widetilde{C}_0 \|L^{-1}\|},$$

then we have

$$\begin{aligned} \|L^{-1} \circ \mathcal{T}_{u_1}(u_3) - L^{-1} \circ \mathcal{T}_{u_1}(u_4)\|_{x^\nu \Lambda^{4,\alpha}} &\leq \frac{3}{8} \|u_3 - u_4\|_{x^\nu \Lambda^{0,\alpha}} \\ &\leq \frac{3}{8} \|u_3 - u_4\|_{x^\nu \Lambda^{4,\alpha}}. \end{aligned}$$

Note that $\|L^{-1}\|$ depends on the projection map P_1 that we construct in Theorem 1.5. Therefore, if L is surjective for $\nu < \frac{n-1}{2}$, and also ϵ and f satisfy the above conditions, then for each $u_1 \in B_\epsilon(0) \cap \text{Ker}(L)$,

$$L^{-1} \circ \mathcal{T}_{u_1} : V_1 \bigcap B_\epsilon(0) \rightarrow V_1 \bigcap B_\epsilon(0)$$

is a contraction map. This implies that there exists a unique $u_2 \in B_\epsilon(0) \cap V_1$, solving the equation

$$L(u_1 + u_2) = \mathcal{T}_{u_1}(u_2).$$

Note that the proof above holds for $h = x^2 g \in C^{4,\alpha}(\overline{M})$. Now we have proved the following theorem,

Theorem 3.1. *Let (M, g) be an asymptotically hyperbolic manifold of dimension $n \geq 4$, with x the smooth defining function, and the metric $h = x^2 g \in C^{4,\alpha}(\overline{M})$. For $0 < \nu < \frac{n-1}{2}$ and $0 < \alpha < 1$, let*

$$L : x^\nu \Lambda^{4,\alpha}(M) \rightarrow x^\nu \Lambda^{0,\alpha}$$

be the linear operator defined in (1.3), which by Theorem 1.5 is essentially surjective. Assume that L is surjective. Then there exists a small constant $\epsilon_0 > 0$, depending on the diameter of M with respect to h , ν , n and also P_1 and L , so that the following holds:

Let ϵ be any small real number satisfying $0 < \epsilon < \epsilon_0$, and let $f \in \Lambda^{0,\alpha}(M)$ satisfy

$$\|Q_g - f\|_{x^\nu \Lambda^{0,\alpha}} \leq \tilde{C} \epsilon,$$

for some positive constant \tilde{C} depending on the diameter of M with respect to h , ν , n , also P_1 and L .

Then for each $u_1 \in B_\epsilon(0) \cap \text{Ker}(L)$, there exists a unique $u \in B_{2\epsilon}(0) \subseteq x^\nu \Lambda^{4,\alpha}(M)$, so that $Q_{\tilde{g}} = f$, where $\tilde{g} = (1 + u)^{\frac{4}{n-4}} g$ for $n \geq 5$, and $\tilde{g} = e^{2u} g$ for $n = 4$, with $P_1 u = u_1$.

By the discussion at the end of Section 2, for the cases in Theorem 1.2 and Theorem 1.3, L is surjective for $x^\nu \Lambda^{4,\alpha}(M)$, $0 < \nu < \frac{n-1}{2}$. This completes the proof of i) of Theorem 1.2 and 1.3.

Since surjectivity is an open property, L is surjective for $x^\nu \Lambda^{4,\alpha}(M)$, $0 < \nu < \frac{n-1}{2}$, for smooth g that is close enough to these metrics. Theorem 3.1 holds for metrics in a small neighborhood of these metrics.

In the following, we will discuss about the boundary regularity of the solutions. For convenience, we assume that the defining function x and the metric $h = x^2 g$ are smooth up to the boundary. The discussion we use here is standard, see [17]. We will sketch the discussion. Composing the inverse G operator of L on both sides of (3.1),

$$(3.8) \quad u - P_1 u = G L u = G \mathcal{T}(u),$$

with $u_1 = P_1 u$ the projection of u to the null space of L .

For the regularity of u with respect to the derivative ∂_y , which is the derivative in some y direction, we introduce the following weighted space with $k \leq m$:

$$x^\nu \Lambda^{m, \alpha, k} = \{ u \in x^\nu \Lambda^{m, \alpha}(M, \sqrt{dx dy}), \text{ so that } (x \partial_x)^j (x \partial_y)^\beta \partial_y^\gamma u \in x^\nu \Lambda^{0, \alpha}, \\ \text{for } j + |\beta| + |\gamma| \leq m, j \geq 0, \text{ and } |\gamma| \leq k. \}.$$

An easy observation is that for $u \in x^\nu \Lambda^{m, \alpha}$ and $m \geq 1$, $\partial_y u = x \partial_y (x^{-1} u)$, so that

$$(3.9) \quad \partial_y u \in x^{\nu-1} \Lambda^{m-1, \alpha}.$$

Also for $u \in x^\nu \Lambda^{m, \alpha, k}$ and $1 \leq k \leq m$, $\partial_y u \in x^\nu \Lambda^{m-1, \alpha, k-1}$. In Proposition 2.9 in [17], it is proved that the inverse operator $G : x^\nu \Lambda^{m, \alpha, k} \rightarrow x^\nu \Lambda^{m+4, \alpha, k}$ is bounded for $m \geq 0$ and $0 \leq k \leq m$; also, $P_1 : x^\nu \Lambda^{m+4, \alpha, k} \rightarrow x^\nu \Lambda^{m+4, \alpha, k}$ is bounded for $m \geq 0$ and $0 \leq k \leq m$.

Lemma 3.2. *Let $u \in x^\nu \Lambda^{4, \alpha}$ be a solution to (3.1) with $1 \leq \nu < \frac{n-1}{2}$ and $0 < \alpha < 1$. Assume that $(f - Q_g) \in x^\nu \Lambda^{m, \alpha, k}$, and $u_1 = P_1 u \in x^\nu \Lambda^{m+4, \alpha, k}$, for $0 \leq k \leq m$. Then we have that $u \in x^\nu \Lambda^{m+4, \alpha, k}$.*

Proof of Lemma 3.2. By assumption, x and the metric h are smooth up to the boundary, so that $Q_g \in C^\infty(\overline{M}) \subseteq \Lambda^{m, \alpha, k}$ for any $m \geq k$, and then we have $f \in \Lambda^{m, \alpha, k}$. For $m = 0$ the claim holds automatically. Now assume $m \geq 1$. Using (3.8) and boundedness of G for $k = 0$ we obtain that $u \in x^\nu \Lambda^{1+4, \alpha}$. Then we can substitute the regularity of u into the right hand side of (3.8), to gain more regularity. Using this induction argument, we obtain $u \in x^\nu \Lambda^{m+4, \alpha} = x^\nu \Lambda^{m+4, \alpha, 0}$. This proves the lemma for $k = 0$.

Define the function F on \mathbb{R} as follows,

$$F(u) = \begin{cases} e^{4u} - 1 - 4u, & n = 4, \\ (1 + u)^{\frac{n+4}{n-4}} - 1 - \frac{n+4}{n-4}u, & n \geq 5. \end{cases}$$

Noticing that for $u \in x^\nu \Lambda^{m, \alpha, k'}$ with $k' < k$, using (3.9) and the fact $\nu \geq 1$, we have that

$$u^2 f = x u (x^{-1} u) f \in x^\nu \Lambda^{m, \alpha, k'+1},$$

raising the third index by 1. This holds for the term $F(u)f$, since F is smooth on \mathbb{R} and vanishes quadratically at 0. Similarly,

$$u(f - Q_g) = x u (x^{-1}(f - Q_g)) = x (x^{-1} u) (f - Q_g) \in x^\nu \Lambda^{m, \alpha, k'+1}.$$

By this fact, combining with the equation (3.8), and also with boundedness of G , an induction argument as the case $k = 0$ proves the Lemma. \square

Now we assume that $f = Q_g$. Generally, $u_1 = P_1 u \in x^\nu \Lambda^{4, \alpha}$ does not have better regularity. In (3.8), the terms on the right hand side behave better than $P_1 u$, and u behaves like $P_1 u$ near the boundary, and u only has the expansion (1.4) with the coefficients which are distributions of negative order, as discussed in Proposition 3.16 in [17]. If $1 \leq \nu < \frac{n-1}{2}$ and $u_1 = P_1 u \in x^\nu \Lambda^{m, \alpha, k}$ for all $m \geq k \geq 0$, which as discussed in [15] is equivalent to say u_1 has a smooth expansion (2.11), then by Lemma 3.2, u has a smooth expansion as in (1.5). Also, for u_1 small enough, we already obtain the existence of u

in Poincaré Einstein manifolds. This completes the proof of Theorem 1.2 and Theorem 1.3. \square

Here we observe that the expansion of u gives us information on the asymptotic behavior of the curvature. For $n = 4$, assume that g and \tilde{g} are asymptotically hyperbolic metrics on M , with the transformation $\tilde{g} = e^{2u}g$, such that u has the expansion $u \sim x^{\frac{3}{2}+i\frac{\sqrt{15}}{2}}u_{00}(y) + x^{\frac{3}{2}-i\frac{\sqrt{15}}{2}}u_{10}(y) + o(x^{\frac{3}{2}})$. Let $(1+v)^2 = e^{2u}$. Denote $\nu_0 = \frac{3}{2} + i\frac{\sqrt{15}}{2}$, and $\nu_1 = \bar{\nu}_0$. Then,

$$\begin{aligned} R_{\tilde{g}} &= (1+v)^{-3}(-6\Delta_g + R_g)(1+v) = e^{-3u}(-6\Delta_g + R_g)e^u \\ &= -6e^{-u}[-3x\partial_x u + (x\partial_x)^2 u] + R_g - 2R_g u + R_g(e^{-2u} - 1 + 2u) \\ &\quad + 6x^2 e^{-3u}(\Delta_y e^u + \frac{1}{2} \sum_{4 \geq i,j \geq 2} h^{ij} \partial_x e^u). \end{aligned}$$

Therefore,

$$\begin{aligned} R_{\tilde{g}} - R_g &= -6e^{-2u}[-3x\partial_x u + (x\partial_x)^2 u] - 2R_g u + R_g(e^{-2u} - 1 + 2u) + 6x^2 e^{-3u}(\Delta_y e^u \\ &\quad + \frac{1}{2} \sum_{4 \geq i,j \geq 2} h_{ij} \partial_x h_{ij} \partial_x e^u) \\ &= -6(-3x\partial_x u + (x\partial_x)^2 u) + 6(1 - e^{-2u})(-3x\partial_x u + (x\partial_x)^2 u) + 24u \\ &\quad - 2u(12 + R_g) + R_g(e^{-2u} - 1 + 2u) + 6x^2 e^{-3u}[\Delta_y e^u + \frac{1}{2} \sum_{4 \geq i,j \geq 2} h^{ij} \partial_x h_{ij} \partial_x e^u] \\ &= -6(-3\nu_0 x^{\nu_0} u_{00}(y) + \nu_0^2 x^{\nu_0} u_{00}(y) - 3\nu_1 x^{\nu_1} u_{10}(y) + \nu_1^2 x^{\nu_1} u_{10}(y) + O(x^{\frac{3}{2}+1})) \\ &\quad + 24(x^{\nu_0} u_{00}(y) + x^{\nu_1} u_{10}(y) + O(x^{\frac{3}{2}+1})) + O(x^{\frac{3}{2}+1}) \\ &= -6((\nu_0^2 - 3\nu_0 - 4)x^{\nu_0} u_{00}(y) + (\nu_1^2 - 3\nu_1 - 4)x^{\nu_1} u_{10}(y)) + O(x^{\frac{3}{2}+1}) \\ &= 120u + o(x^{\frac{3}{2}}) \end{aligned}$$

For asymptotically hyperbolic manifolds of higher dimension, with similar calculation, we obtain the formula

$$R_{\tilde{g}} - R_g = \frac{4(n-1)(n^2 + 2n - 4)}{(n-4)}u + o(x^{\frac{n-1}{2}}).$$

4. CONSTANT Q -CURVATURE METRICS FOR PERTURBED CONFORMAL STRUCTURES

Let (M, g_0) be a Poincaré-Einstein manifold, with a defining function x and the metric $h_0 = x^2 g_0$ smooth up to the boundary. Let

$$\begin{aligned} \mathfrak{M}_\tau &= \{h : \text{metrics on } \overline{M}, \text{ so that } h \in C^{4,\alpha}(\overline{M}), \\ &\quad \text{with } \|h - h_0\|_{C^{4,\alpha}(M)} \leq \tau, \text{ and } |dx|_h|_{\partial M} = 1\}, \end{aligned}$$

for $\tau > 0$ and $0 < \alpha < 1$. For $h \in \mathfrak{M}_\tau$, let $g = x^{-2}h$. We want to see that if τ is small enough, whether we can find a constant Q -curvature metric \tilde{g} in the conformal class of g , with $Q_{\tilde{g}} = Q_{g_0}$. We use the same notation u , L_g and so on as above. Note that the choice of x that $|dx|_h = 1$ in the sections before is only to make the notation simpler.

Now we only assume that $|dx|_h = 1$ on ∂M , and then there are only some additional small terms in $E(L)$. It is easy to check that

$$x^\alpha \Lambda^{0,\alpha}(M, \sqrt{dx dy}) = \{u \in C^\alpha(\overline{M}), u|_{\partial M} = 0\}.$$

Let L_g and L_{g_0} be the linear operators (1.3) with respect to g and g_0 . Recall that Ric_g and R_g satisfy (2.2) and (2.3). We know that

$$(|dx|_h^2 - 1) \in x^\alpha \Lambda^{0,\alpha}(M, \sqrt{dx dy}), \text{ and } \| |dx|_h^2 - |dx|_{h_0}^2 \|_{x^\alpha \Lambda^{0,\alpha}} \leq C \tau,$$

for some constant C depending on the defining function and h_0 . Also it is easy to see the following inequalities by the formula of the coefficients

$$\begin{aligned} \|(\Delta_g^2 - \Delta_{g_0}^2)u\|_{x^\alpha \Lambda^{0,\alpha}} &\leq C \tau \|u\|_{x^\alpha \Lambda^{4,\alpha}}, \\ \|(R_g \Delta_g - R_{g_0} \Delta_{g_0})u\|_{x^\alpha \Lambda^{0,\alpha}} &\leq C \tau \|u\|_{x^\alpha \Lambda^{4,\alpha}}, \\ \|(\text{Ric}_{ij}(g) \nabla_g^i \nabla_g^j - \text{Ric}_{ij}(g_0) \nabla_{g_0}^i \nabla_{g_0}^j)u\|_{x^\alpha \Lambda^{0,\alpha}} &\leq C \tau \|u\|_{x^\alpha \Lambda^{4,\alpha}}, \\ \|(\nabla_g R_g, \nabla_g u) - (\nabla_{g_0} R_{g_0}, \nabla_{g_0} u)\|_{x^\alpha \Lambda^{0,\alpha}} &\leq C \tau \|u\|_{x^\alpha \Lambda^{4,\alpha}}, \\ \|Q_g - Q_{g_0}\|_{x^\alpha \Lambda^{0,\alpha}} &\leq C \tau, \end{aligned}$$

with C depending on the defining function x and the metric h_0 . We know that L_{g_0} is surjective. Let

$$(4.1) \quad x^\alpha \Lambda^{4,\alpha}(M, \sqrt{dx dy}) = \text{Ker}(L_{g_0}) \oplus V_1(g_0),$$

be the splitting as in Theorem 1.5. Restricted on V_1 with respect to g_0 , L_{g_0} satisfies (3.2). Therefore, we can choose $\tau > 0$ small enough so that $\|L_g - L_{g_0}\| \leq \frac{1}{2} C_1$ with C_1 in (3.2). Then we have that

$$(4.2) \quad \frac{1}{2} C_1 \|u\|_{x^\alpha \Lambda^{4,\alpha}} \leq \|L_g u\|_{x^\alpha \Lambda^{0,\alpha}} \leq (C_2 + \frac{1}{2} C_1) \|u\|_{x^\alpha \Lambda^{4,\alpha}},$$

for $u \in V_1(g_0)$. Then $L_g : V_1(g_0) \rightarrow x^\alpha \Lambda^{0,\alpha}(M, \sqrt{dx dy})$ is isomorphic so that

$$(4.3) \quad \|L_g^{-1}\| \leq \frac{2}{C_1}.$$

and then $\text{Ker}(L_g) \subseteq \text{Ker}(L_{g_0})$. We will only use the splitting of the weighted space with respect to g_0 . Now we have a uniform constant $\epsilon > 0$ for all $h \in \mathfrak{M}_\tau$ and $g = x^2 h$ so that it satisfies the conditions (3.4) and (3.6). Furthermore, we assume that $\tau > 0$ is small enough so that

$$(4.4) \quad \|Q_g - Q_{g_0}\|_{x^\alpha \Lambda^{0,\alpha}} \leq C \tau,$$

and it satisfies corresponding inequalities as (3.5) and (3.7). Therefore, the proof of Theorem 3.1 applies. We then obtain the following perturbation result.

Theorem 4.1. *Let (M, g_0) be a Poincaré-Einstein manifold with defining function x and the metric $h_0 = x^2 g_0$ smooth up to the boundary, and let \mathfrak{M}_τ be as above, with $\tau > 0$. There exists $\tau_0 > 0$, so that for $0 < \tau < \tau_0$, and any metric $h \in \mathfrak{M}_\tau$, there always exist a family of asymptotically hyperbolic metrics in the conformal class of $g = x^{-2} h$ with constant Q -curvature Q_{g_0} , which are parametrized by elements in $\text{Ker}(L_{g_0})$.*

5. CRITICAL METRICS OF REGULARIZED DETERMINANTS

Let M be a fourth dimensional asymptotically hyperbolic manifold, with complete metric g and its smooth defining function x , so that $h = x^2 g$ is a smooth metric on \overline{M} . Consider the equation

$$(5.1) \quad U = U_g \equiv \gamma_1 |W|^2 + \gamma_2 Q - \gamma_3 \Delta R = C,$$

where $\gamma_1, \gamma_2, \gamma_3$ and C are some constants, W is the Weyl tensor, and Q, R the Q -curvature and the scalar curvature with respect to g . The equation arises as the Euler-Lagrange equation for the regularized determinants,

$$F_A[w] = \log\left(\frac{\det A_{\tilde{g}}}{\det A_g}\right),$$

of a conformally covariant operator $A = A_g$, under the conformal change of metrics $\tilde{g} = e^{2w} g$, see Chapter 6 in [12]. More precisely, under the conformal change,

$$(5.2) \quad \tilde{U} e^{4w} = U + \left(\frac{1}{2}\gamma_2 + 6\gamma_3\right)\Delta^2 w + 6\gamma_3 \Delta |\nabla w|^2 - 12\gamma_3 \nabla^i [(\Delta w + |\nabla w|^2) \nabla_i w]$$

$$(5.3) \quad + \gamma_2 R_{ij} \nabla_i \nabla_j w + (2\gamma_3 - \frac{1}{3}\gamma_2) R \Delta w + (2\gamma_3 + \frac{1}{6}\gamma_2) (\nabla R, \nabla w),$$

with $\tilde{U} = U_{\tilde{g}}$. Define $\alpha = \frac{\gamma_2}{12\gamma_3}$. The following are some examples that we are interested in.

Example 1. For the conformal Laplacian, $A = L$, we have that $(\gamma_1, \gamma_2, \gamma_3) = (1, -4, -\frac{2}{3})$, and $\alpha = \frac{1}{2}$.

Example 2. For the spin Laplacian, $A = D^2$, we have that $(\gamma_1, \gamma_2, \gamma_3) = (7, -88, -\frac{14}{3})$, and $\alpha = \frac{11}{7}$.

Example 3. For the Paneitz operator, $A = P$, we have that $(\gamma_1, \gamma_2, \gamma_3) = (-\frac{1}{4}, -14, \frac{8}{3})$, and $\alpha = \frac{-7}{16}$.

For convenience, dividing both sides of the function by $6\gamma_3$, we have the following equation,

$$(5.4) \quad \frac{\tilde{U}}{6\gamma_3} e^{4w} = (1 + \alpha) \Delta^2 w + \Delta |\nabla w|^2 - 2 \nabla^i [(\Delta w + |\nabla w|^2) \nabla_i w] + 2\alpha R_{ij} \nabla^i \nabla^j w$$

$$(5.5) \quad + \left(\frac{1}{3} - \frac{2}{3}\alpha\right) R \Delta w + \left(\frac{1}{3} + \frac{1}{3}\alpha\right) (\nabla R, \nabla w) + \frac{U}{6\gamma_3}.$$

We should note that

$$\begin{aligned} \Delta |\nabla w|^2 - 2 \nabla^i (\Delta w \nabla_i w) &= 2(\Delta_g \nabla w, \nabla w) + 2(\nabla^2 w, \nabla^2 w) - 2 \nabla^i (\Delta w \nabla_i w) \\ &= 2 \nabla^i w (g^{pq} \nabla_p \nabla_q \nabla_i w - g^{pq} \nabla_i \nabla_p \nabla_q w) + 2(|\nabla^2 w|_g^2 - (\Delta w)^2) \\ &= 2 \nabla^i w g^{pq} R_{piq}^s \nabla_s w + 2(|\nabla^2 w|_g^2 - (\Delta w)^2) \\ &= 2 Ric(\nabla w, \nabla w) + 2(|\nabla^2 w|_g^2 - (\Delta w)^2). \end{aligned}$$

Moreover,

$$\nabla^i(|\nabla w|^2 \nabla_i w) = 2 \nabla^i \nabla^j w \nabla_j w \nabla_i w + |\nabla w|^2 \Delta w,$$

therefore, the equation can be written in the following way,

$$(5.6) \quad \frac{\tilde{U}}{6\gamma_3} e^{4w} = (1 + \alpha) \Delta^2 w + 2 \operatorname{Ric}(\nabla w, \nabla w) + 2(|\nabla^2 w|_g^2 - (\Delta w)^2)$$

$$(5.7) \quad - 4 \nabla_i \nabla_j w \nabla_j w \nabla_i w - 2 |\nabla w|^2 \Delta w + 2\alpha R_{ij} \nabla^i \nabla^j w$$

$$(5.8) \quad + \left(\frac{1}{3} - \frac{2}{3}\alpha\right) R \Delta w + \left(\frac{1}{3} + \frac{1}{3}\alpha\right) (\nabla R, \nabla w) + \frac{U}{6\gamma_3}.$$

We should point out that for $\alpha = -1$ and $\gamma_1 = 0$, the equation reduces to a second order differential equation, and in this case the U -curvature relates to the σ_2 -curvature with respect to the Schouten tensor $A(g)$,

$$\begin{aligned} \frac{1}{12\gamma_3} U(g) &= \frac{\gamma_1}{12\gamma_3} |W|_g^2 + \frac{\gamma_2}{12\gamma_3} Q_g - \frac{\Delta R_g}{12} \\ &= -\left(\frac{-1}{4} |\operatorname{Ric}_g|^2 + \frac{1}{12} R_g^2 - \frac{1}{12} \Delta_g R_g\right) - \frac{\Delta R_g}{12} \\ &= -\left(\frac{-1}{4} |\operatorname{Ric}_g|^2 + \frac{1}{12} R_g^2\right) = -2\sigma_2(g). \end{aligned}$$

We have the equation

$$\begin{aligned} 4\sigma_2(\tilde{g}) &= -2\operatorname{Ric}(\nabla_g w, \nabla_g w) - 2(|\nabla^2 w|_g^2 - (\Delta w)^2) + 4 \nabla_{ij} w \nabla_i w \nabla_j w \\ &\quad + 2|\nabla w|^2 \Delta w + 2\operatorname{Ric}_{ij} \nabla^i \nabla^j w - R_g \Delta w + 4\sigma_2(g). \end{aligned}$$

A prescribed constant σ_2 -curvature asymptotically hyperbolic metric problem is discussed in [19]. From now on, we assume that $\alpha \neq -1$.

The linearization of (5.6) is given by

$$L w = (1 + \alpha) \Delta^2 w + 2\alpha R_{ij} \nabla^i \nabla^j w + \left(\frac{1}{3} - \frac{2}{3}\alpha\right) R \Delta w + \left(\frac{1}{3} + \frac{1}{3}\alpha\right) (\nabla R, \nabla w) - \frac{2U}{3\gamma_3} w = 0.$$

As $x \rightarrow 0$,

$$\begin{aligned} R_{ijkl}(g) &= x^{-2} [R_{ijkl}(h) - h_{ik}(x^{-1} \nabla_j^h \nabla_l x + \frac{1}{2} x^{-2} h_{jl}) - h_{jl}(-x^{-1} \nabla_i^h \nabla_k x + \frac{1}{2} x^{-2} h_{ik}) \\ &\quad + h_{il}(-x^{-1} \nabla_j^h \nabla_k x + \frac{1}{2} x^{-2} h_{jk}) + h_{jk}(-x^{-1} \nabla_i^h \nabla_l x + \frac{1}{2} x^{-2} h_{il})] \\ &= x^{-4} \left[-\frac{1}{2} h_{ik} h_{jl} - \frac{1}{2} h_{jl} h_{ik} + h_{il} h_{jk} + \frac{1}{2} h_{jk} h_{il} + O(x)\right] \\ &= x^{-4} [-h_{ik} h_{jl} + h_{il} h_{jk} + O(x)], \end{aligned}$$

while

$$A(g) = \frac{1}{4-2} (\operatorname{Ric}(g) - \frac{1}{2(4-1)} R(g)g) = \frac{1}{2} (-3 + 2 + O(x))g = \left(-\frac{1}{2} + O(x)\right)g,$$

so that

$$\begin{aligned}
W_{ijkl}(g) &= R_{ijkl}(g) - g_{ik}A_{jl}(g) + g_{il}A_{jk}(g) + g_{jk}A_{il}(g) - g_{jl}A_{ik}(g) \\
&= x^{-4}(-h_{ik}h_{jl} + h_{il}h_{jk} + O(x)) + x^{-4}[-h_{ik}(-\frac{1}{2}h_{jl} + O(x)) \\
&\quad + h_{il}(-\frac{1}{2}h_{jk} + O(x)) + h_{jk}(-\frac{1}{2}h_{il} + O(x)) - h_{jl}(-\frac{1}{2}h_{ik} + O(x))] \\
&= x^{-4}O(x),
\end{aligned}$$

and moreover, using the fact $\Delta_h R = O(x)$, and $Q(g) = 3 + O(x)$, we have that $U(g) = 3\gamma_2 + O(x)$. Then we obtain the main terms of Lw as follows,

$$\begin{aligned}
Lw &= (1 + \alpha)\Delta_g^2 w + (\frac{1}{3} - \frac{2}{3}\alpha)R_g\Delta_g w + 2\alpha \text{Ric}_{ij}^g \nabla_g^i \nabla_g^j w \\
&\quad + \frac{1}{3}(1 + \alpha)(\nabla_g R_g, \nabla_g w) - \frac{2U}{3\gamma_3}w \\
&= (1 + \alpha)\Delta_g^2 w + (\frac{1}{3} - \frac{2}{3}\alpha)(-12 + O(x))\Delta_g w + 2\alpha(-3\Delta_g w + O(x)p(x, y, x\partial_x, x\partial_y)w) \\
&\quad + \frac{1}{3}(1 + \alpha)((-2 \times 4 - 2)x^2 H(h|_{s_x})\partial_x w + O(x^3)|\nabla_y w|) - (8 \times 3\alpha + O(x))w \\
&= (1 + \alpha)\Delta_g^2 w - 12(\frac{1}{3} - \frac{2}{3}\alpha)\Delta_g w - 6\alpha\Delta_g w - 24\alpha w + O(x)p(x, y, x\partial_x, x\partial_y)w \\
&= (1 + \alpha)\Delta_g^2 w - (4 - 2\alpha)\Delta_g w - 24\alpha w + O(x)p(x, y, x\partial_x, x\partial_y)w \\
&= ((1 + \alpha)\Delta_g + 6\alpha)(\Delta_g - 4)w + O(x)p(x, y, x\partial_x, x\partial_y)w.
\end{aligned}$$

Correspondingly,

$$N(L)w = (1 + \alpha)((s\partial_s)^2 + s^2\Delta_v - 3s\partial_s)^2 w + (2\alpha - 4)((s\partial_s)^2 + s^2\Delta_v - 3s\partial_s)w - 24\alpha w,$$

$$\begin{aligned}
L_0(t, \hat{\eta})w &= (1 + \alpha)((t\partial_t)^2 + t^2 - 3t\partial_t)^2 w + (2\alpha - 4)((t\partial_t)^2 + t^2 - 3t\partial_t)w - 24\alpha w \\
&= ((1 + \alpha)((t\partial_t)^2 + t^2 - 3t\partial_t) + 6\alpha)((t\partial_t)^2 + t^2 - 3t\partial_t - 4)w = L_3 \circ L_1 w,
\end{aligned}$$

$$\begin{aligned}
I(L)w &= (1 + \alpha)((s\partial_s)^2 - 3s\partial_s)^2 w + (2\alpha - 4)((s\partial_s)^2 - 3s\partial_s)w - 24\alpha w \\
&= ((1 + \alpha)((s\partial_s)^2 - 3s\partial_s + 6\alpha)((s\partial_s)^2 - 3s\partial_s - 4)w.
\end{aligned}$$

Therefore, the indicial roots of L is as follows,

- i) For $\alpha = \frac{1}{2}$, $\text{spec}_b(L) = \{4, -1, 1, 2\}$.
- ii) For $\alpha = \frac{11}{7}$, $\text{spec}_b(L) = \{4, -1, \frac{3}{2} + i\frac{\sqrt{51}}{6}, \frac{3}{2} - i\frac{\sqrt{51}}{6}\}$.
- iii) For $\alpha = \frac{-7}{16}$, $\text{spec}_b(L) = \{4, -1, \frac{3}{2} + \frac{\sqrt{249}}{6}, \frac{3}{2} - \frac{\sqrt{249}}{6}\}$.

The solution of $L_1 w = 0$ is exactly the same as discussed in Section 2. We solve $L_3 w = 0$ by transferring it into the Bessel type equations discussed as above. Let $u(t) = t^\beta \tilde{w}(t)$, then

$$0 = t^\beta((t\partial_t)^2 \tilde{w} + (2\beta - 3)t\partial_t \tilde{w} + (\beta^2 - 3\beta + \frac{6\alpha}{1 + \alpha} - t^2)\tilde{w}).$$

Let $2\beta - 3 = 0$, and then $\beta = \frac{3}{2}$. Consequently,

$$[(t\partial_t)^2 - (t^2 + \frac{9}{4} - \frac{6\alpha}{1+\alpha})]\tilde{w} = 0.$$

Let $\tilde{\alpha}^2 = \frac{9}{4} - \frac{6\alpha}{1+\alpha}$, then the solution is

$$(5.9) \quad w = t^{\frac{3}{2}}(C_1 I_{\tilde{\alpha}}(t) + C_2 K_{\tilde{\alpha}}(t)).$$

Here $\tilde{\alpha}^2$ is $\frac{1}{4}$, $\frac{-17}{12}$, $\frac{83}{12}$, corresponding to the above three cases, with $\text{Re}(\tilde{\alpha}) \geq 0$. For the case $\tilde{\alpha}^2 = -\frac{17}{12}$, since $\tilde{\alpha}^2$ is negative, L_3 behaves the same as L_2 in Section 2, and it follows that Theorem 1.4 and Theorem 1.5 with $n = 4$ hold for the linear operator L , using the same argument as in Section 2.

By the expansion of the series form of the Bessel functions, as in [[13], P. 108], we have

$$t^{\frac{3}{2}} I_{\tilde{\alpha}}(t|\eta|) \sim t^{\frac{3}{2}+\tilde{\alpha}}|\eta|^{\tilde{\alpha}}/(2^{\tilde{\alpha}}\Gamma(1+\tilde{\alpha})),$$

and

$$t^{\frac{3}{2}} I_{-\tilde{\alpha}}(t|\eta|) \sim t^{\frac{3}{2}-\tilde{\alpha}}|\eta|^{-\tilde{\alpha}}/(2^{-\tilde{\alpha}}\Gamma(1-\tilde{\alpha})),$$

near $t = 0$. Here we should note that the series expansion applies for all $\tilde{\alpha} \in \mathbb{C}$. Now it is easy to see that the linear combination

$$x^{\frac{3}{2}}(C_1 x^{\tilde{\alpha}} + C_2 x^{-\tilde{\alpha}})$$

can never vanish to infinite order at $t = 0$ if either $C_1 \neq 0$ or $C_2 \neq 0$. Also,

$$t^{\frac{3}{2}} K_{\tilde{\alpha}}(t|\eta|) \sim t^{\frac{3}{2}} \frac{\pi I_{\tilde{\alpha}}(t|\eta|) - I_{-\tilde{\alpha}}(t|\eta|)}{\sin(\tilde{\alpha}\pi)} \sim O((t|\eta|)^{\frac{3}{2}-\tilde{\alpha}}),$$

near $t = 0$, with $\tilde{\alpha} > 0$ and $\tilde{\alpha} \neq 1, 2, 3, \dots$

Using the integral form, we have

$$t^{\frac{3}{2}} I_{\tilde{\alpha}}(t|\eta|) \text{ grows exponentially, } t^{\frac{3}{2}} K_{\tilde{\alpha}}(t|\eta|) \text{ decays exponentially}$$

near $t = +\infty$. Therefore, $t^{\frac{3}{2}} I_{\tilde{\alpha}}(t|\eta|)$ does not belong to $t^{\delta} L^2(\mathbb{R}^+)$ for any $\delta > 0$, while

$$t^{\frac{3}{2}} K_{\tilde{\alpha}}(t|\eta|) \in t^{\delta} L^2(\mathbb{R}_+),$$

only for $\delta < \frac{3}{2} + \frac{1}{2} - \tilde{\alpha} = 2 - \tilde{\alpha}$. That is, L_3 is injective in $x^{\delta} L^2$ for $\delta > 2 - \tilde{\alpha}$.

Summarizing the above discussion, let us compute $\bar{\delta}$ and $\underline{\delta}$ for the linearized operator L .

$$\bar{\delta} = \inf \{ \delta : L_1 \text{ and } L_3 \text{ are injective in } t^{\delta} L^2 \} = \sup \{ -1 + \frac{1}{2}, 2 - \tilde{\alpha} \}, \text{ and dually,}$$

$$\underline{\delta} = \inf \{ (\frac{3}{2} + \frac{1}{2}) \times 2 - (-1 + \frac{1}{2}), (\frac{3}{2} + \frac{1}{2}) \times 2 - (2 - \tilde{\alpha}) \} = \inf \{ \frac{9}{2}, 2 + \tilde{\alpha} \}.$$

For the case $\alpha = \frac{1}{2}$, $\bar{\delta} = \frac{3}{2}$, and $\underline{\delta} = \frac{5}{2}$ (surjectivity). For the case $\alpha = -\frac{7}{16}$, $\bar{\delta} = -1 + \frac{1}{2} = -\frac{1}{2}$, and $\underline{\delta} = \frac{9}{2}$. Then we can use Theorem 1.4 and Theorem 1.5, to obtain the semi-Fredholm property for these linear operators.

For the Poincaré Einstein manifold (M, g) , we have that the U curvatures defined above are all constants on M . We want to see the solutions of the nonlinear problem. Now

$Lw = ((1 + \alpha)\Delta_g + 6\alpha)(\Delta_g - 4)w$. Define the operator $\mathcal{T} : x^\nu \Lambda^{4,\alpha}(M) \rightarrow x^\nu \Lambda^{0,\alpha}(M)$ as follows,

$$\begin{aligned} \mathcal{T}(w) = & \left(\frac{\tilde{U}}{6\gamma_3} e^{4w} - \frac{U}{6\gamma_3} - \frac{2}{3\gamma_3} U w \right) - 2\text{Ric}(\nabla w, \nabla w) \\ & - 2(|\nabla^2 w|_g^2 - (\Delta w)^2) + 4\nabla_j \nabla_i w \nabla^j w \nabla^i w + 2|\nabla w|^2 \Delta w. \end{aligned}$$

We rewrite it in the form

$$\begin{aligned} \mathcal{T}(w) = & \frac{\tilde{U}}{6\gamma_3} (e^{4w} - 1 - 4w) + (\tilde{U} - U) \left(\frac{1}{6\gamma_3} + \frac{2}{3\gamma_3} w \right) - 2\text{Ric}(\nabla w, \nabla w) \\ & - 2(|\nabla^2 w|_g^2 - (\Delta w)^2) + 4\nabla_j \nabla_i w \nabla^j w \nabla^i w + 2|\nabla w|^2 \Delta w. \end{aligned}$$

In this formula, comparing with the nonlinear term defined for Q -curvature equation, a few square terms of w and its derivatives of order up to 2 are involved, which are small terms in the argument of the perturbation problem. Now, the nonlinear equation becomes

$$L_g w = \mathcal{T}(w).$$

To solve this, the argument follows exactly the way in Section 3 and Section 4. We only need to choose the right weighted Hölder spaces. Note that the index of the weight for the Hölder space is $\frac{1}{2}$ less than the index of the weight of the corresponding Sobolev spaces.

5.1. Summary. Perturbation results for the curvatures defined in (5.1) can be proved along the same lines as the Q -curvature. For instance, assume (M, g) is a Poincaré-Einstein manifold. For the case $\alpha = -\frac{7}{16}$, by maximal principle, $((1 + \alpha)\Delta_g + 6\alpha)$ and $(\Delta_g - 4)$ are both injective on $L^2(M, g)$. Then similar to the discussion for the Q -curvature equation, there are infinitely many solutions $u \in x^\nu \Lambda^{4,\beta}(M, \sqrt{dx dy})$ for $0 < \beta < 1$ to this equation parametrized by the projection $P_1 u$ to the kernel of the linearized operator L , for $\nu \in (0, \frac{3}{2})$. Moreover, if $\tilde{U} = U$, then w has the weak expansion $w(x, y) \sim w_{00}(y)x^4 + o(x^4)$, and also w has a smooth expansion if $1 \leq \nu < \frac{3}{2}$ and $P_1 w$ has a smooth expansion. For the case $\alpha = \frac{11}{7}$, it is the same as the Q curvature problem, and the only difference is that here we use $i\sqrt{51}$ in the indicial roots and in the formula of expansion to replace $i\sqrt{15}$. For the case $\alpha = \frac{1}{2}$, $((1 + \alpha)\Delta_g + 6\alpha)$ is essentially injective on $x^\nu \Lambda^{4,\beta}(M, \sqrt{dx dy})$ for $\nu > 1$ and $\nu \neq 2$, while it is essentially surjective on $x^\nu \Lambda^{4,\beta}(M, \sqrt{dx dy})$ for $\nu < 2$, also $\nu \neq 1$ and $0 < \beta < 1$. Since $(\frac{3}{2}\Delta_g + 3)$ may have finite dimensional kernel, we do not have perturbation result for ν in this interval. But note that, using the same argument as in Lemma 2.6 in weighted Hölder spaces, for $\nu > 2$, the operator

$$\left(\frac{3}{2}\Delta + 3\right) : x^\nu \Lambda^{2+m,\beta} \rightarrow x^\nu \Lambda^{m,\beta},$$

is injective, for $0 < \beta < 1$ and $m \geq 0$. Then dually the operator $(\frac{3}{2}\Delta + 3)$ is surjective for $\nu \in (0, 1)$. Also we know that the operator $(\Delta_g - 4)$ is surjective in the weighted Hölder space with $0 < \nu < \frac{3}{2}$, then the linearized operator

$$L : x^\nu \Lambda^{4+m,\beta} \rightarrow x^\nu \Lambda^{m,\beta},$$

with $m \geq 0$ is surjective for $0 < \nu < 1$ and $0 < \beta < 1$. Therefore, for the case $\alpha = \frac{1}{2}$, the existence result as in i) in Theorem 1.3 holds for $0 < \nu < 1$. For the boundary expansion when $\tilde{U} = U$, since all the indicial roots are integers in this case, there may be $\log(x)$ terms in the expansion as (2.10). Also, since $\nu < 1$, the smooth expansion result does not hold.

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